Name: Date: 03/30/2023

M20580 L.A. and D.E. Tutorial
Worksheet 10

1. Determine whether the statements are true or false and justify your answer.

   (a) Every linearly independent set of vectors are orthogonal.

   (b) Every orthogonal set of vectors are linearly independent.

   (c) Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

   (d) $\text{proj}_W \mathbf{x}$ is orthogonal to every vector in W.

   (e) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then $A^T A\mathbf{x} = A^T\mathbf{b}$ is also consistent.

   (f) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then $A^T A\mathbf{x} = A^T\mathbf{b}$ is also inconsistent.

Solution:

(a) False. In $\mathbb{R}^2$, the vectors $(1, 0)$ and $(1, 1)$ form basis, hence they are linearly independent, but they are not orthogonal since $(1, 0) \cdot (1, 1) = 1 \neq 0$.

(b) False. Any orthogonal set of **nonzero** vectors are linearly independent.

(c) True. Any nontrivial subspace has a basis and we can use the Gram-Schmidt process to find an orthonormal basis.

(d) False. W contains the vector $\text{proj}_W \mathbf{x}$. What is true is that $\mathbf{x} - \text{proj}_W \mathbf{x}$ is orthogonal to every vector in W.

(e) True. A solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$. If $A\mathbf{x} = \mathbf{b}$ is consistent, then multiplying both sides on the left by A^T gives $A^T A\mathbf{x} = A^T\mathbf{b}$. In other words, any solution to $A\mathbf{x} = \mathbf{b}$ is aslo a solution to $A^T A\mathbf{x} = A^T\mathbf{b}$.

(f) False. $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent. A solution is a least squares solution to $A\mathbf{x} = \mathbf{b}$.

2. Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ onto the subspace $W = \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)$ in $\mathbb{R}^3$, where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Find the distance from $\mathbf{v}$ to W.

   Is there a fast way to verify that the distance from $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to W is 0?

Solution: Note that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, thus $\mathbf{u}_1$ and $\mathbf{u}_2$ form orthogonal basis of W. Then

$$\operatorname{proj}_W(\mathbf{v}) = \frac{(1,2,-3)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) + \frac{(1,2,-3)\cdot(-1,2,1)}{\|(-1,2,1)\|^2}(-1,2,1)$$

$$\operatorname{proj}_W(\mathbf{v}) = (-1,0,-1)$$

$$\operatorname{perp}_W(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = (1,2,-3) - (-1,0,-1) = (0,2,-2)$$

Distance from $\mathbf{v}$ to W is $\|\operatorname{perp}_W(\mathbf{v})\| = 2\sqrt{2}$. Yes, $\mathbf{x}$ is linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ thus it is in W and distance from it to W is 0.

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3. Find the standard matrix, P, of the orthogonal projection onto the subspace $W = \operatorname{span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$. Then use this matrix to find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ onto W.

Solution: Let (x, y) be a vector in $\mathbb{R}^2$. Then $\operatorname{proj}_W((x, y)) = \frac{(x,y)\cdot(3,-1)}{\|(3,-1)\|^2}(3, -1) = \frac{3x-y}{10}(3, -1)$. Since $\operatorname{proj}_W((1, 0)) = (9/10, -3/10)$ and $\operatorname{proj}_W((0, 1)) = (-3/10, 1/10)$, then $P = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$.

Using the matrix P, we find the orthogonal projection of $\mathbf{v} = (-1, 1)$ onto W is $P = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6/5 \\ 2/5 \end{bmatrix}$.

Alternative solution:
Note that $W = \operatorname{span}((3, -1))$ is the also span of the $(3/10, -1/10)$ which form orthonormal basis of W. Then the matrix of orthogonal projection can be find as: $\begin{bmatrix} \frac{3}{10} \\ \frac{-1}{10} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{10} & \frac{-1}{10} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{1}{10} \end{bmatrix}$.

4. Use the Gram-Schmidt Process to find an orthogonal basis for the column space of the matrix $\begin{bmatrix} 0 & -2 & -2 \\ 1 & 0 & 0 \\ 1 & -2 & -1 \end{bmatrix}$. Normalize the obtained orthogonal basis.

Solution: Let $v_1 = (0,1,1)$.

Then $v_2 = (-2,0,-2) - \frac{(-2,0,-2)\cdot(0,1,1)}{\|(0,1,1)\|^2}(0,1,1) = (-2,1,3)$.

Then $v_3 = (-2,0,-1) - \left(\frac{(-2,0,-1)\cdot(0,1,1)}{\|(0,1,1)\|^2}(0,1,1) + \frac{(-2,0,-1)\cdot(-2,1,3)}{\|(-2,1,3)\|^2}(-2,1,3)\right)$
$= (-13/7, 3/7, 9/7)$.

After normalization we have:

$u_1 = \frac{v_1}{\|v_1\|} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$,

$u_2 = \frac{v_2}{\|v_2\|} = (-2/\sqrt{14},\ 1/\sqrt{14},\ 3/\sqrt{14})$,

$u_3 = \frac{v_3}{\|v_3\|} = (-13/\sqrt{259}, 3/\sqrt{259}, 9/\sqrt{259})$,

5. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, by using normal equations.

   Hint:
   (i) Observe that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

   (ii) Let $\hat{\mathbf{x}}$ be the least squares solution of $A\mathbf{x} = \mathbf{b}$. Recall that $\mathbf{b} - A\hat{\mathbf{x}} = \mathrm{perp}_{\mathrm{Col(A)}}(\mathbf{b})$. The least square solution $\hat{\mathbf{x}}$ satisfies $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$ or $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

   (iii) Find A^TA and $A^T\mathbf{b}$. Solve $A^TA\mathbf{x} = A^T\mathbf{b}$ to find $\hat{\mathbf{x}}$.

Solution: We have $A^T = \begin{bmatrix} 1 & 0 & 2 \\ -2 & -3 & 5 \end{bmatrix}$. So, $A^TA = \begin{bmatrix} 5 & 8 \\ 8 & 38 \end{bmatrix}$, and $A^T\mathbf{b} = \begin{bmatrix} 0 \\ -21 \end{bmatrix}$. So, the normal equation is $\begin{bmatrix} 5 & 8 \\ 8 & 38 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ -21 \end{bmatrix}$.

Thus,
$$\mathbf{x} = \begin{bmatrix} 38/126 & -8/126 \\ -8/126 & 5/126 \end{bmatrix} \begin{bmatrix} 0 \\ -21 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -5/6 \end{bmatrix}$$
is a least squares solution.