Name: SOLUTIONS Date: 04/06/2023

M20580 L.A. and D.E. Tutorial
Worksheet 11

1. Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution: First we use Gram-Schmidt process to produce an orthogonal set.

$$\begin{aligned}
\mathbf{v_1} &= \mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{v_3} &= \mathbf{x_3} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_3}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} - \left(\frac{\mathbf{v_2} \cdot \mathbf{x_3}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right) \mathbf{v_2} \\
&= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\end{aligned}$$

Then the orthonormal basis for col(A) is

$$\left\{ \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}, \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|}, \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} \right\} = \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}.$$

$A = QR$ for some upper triangular matrix R, to find R we use the fact that Q has orthonormal columns, hence $Q^T Q = I$. Therefore $Q^T A = Q^T QR = IR = R$

$$R = Q^T A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & -2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

2. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix}$ by using normal equations.

Solution: We have $A^T = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix}$. So $A^T A = \begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix}$, and $A^T\mathbf{b} = \begin{bmatrix} 12 \\ -24 \end{bmatrix}$. Then the normal equation is

$$\begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 12 \\ -24 \end{bmatrix}$$

Thus

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T\mathbf{b} = \begin{bmatrix} 19/234 & -2/117 \\ -2/117 & 7/234 \end{bmatrix} \begin{bmatrix} 12 \\ -24 \end{bmatrix} = \begin{bmatrix} 18/13 \\ -12/13 \end{bmatrix}$$

is a least squares solution.

3. For each of the following, state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6) in Section 1.1:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

(a) $(1-x)y'' - 4xy' + 5y = \cos(x)$

(b) $t^5 y^{(4)} - t^3 y'' + 6y = 0$

(c) $\frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2}$

(d) $(\sin\theta)y''' - (\cos\theta)y' = 2$

Solution: (a) The highest derivative in the equation is y''; thus the equation is second order. Matching with (6) in Section 1.1 gives $a_2(x) = 1 - x$, $a_1(x) = -4x$, $a_0(x) = 5$, and $g(x) = \cos(x)$; thus the equation is linear.

(b) The highest derivative in the equation is $y^{(4)}$; thus the equation is fourth order. Matching with (6) in Section 1.1 gives $a_4(t) = t^5$, $a_2(t) = -t^3$, $a_0(t) = 6$, and $a_3(t) = a_1(t) = g(t) = 0$; thus the equation is linear.

(c) The highest derivative in the equation is $\frac{d^2y}{dx^2}$; thus the equation is second order. Matching with (6) in Section 1.1 is not possible; thus the equation is nonlinear.

(d) The highest derivative in the equation is y'''; thus the equation is third order. Matching with (6) in Section 1.1 gives $a_3(\theta) = \sin(\theta)$, $a_1(\theta) = -\cos(\theta)$, $g(\theta) = 2$, and $a_2(\theta) = a_0(\theta) = 0$; thus the equation is linear.

4. For each of the following, verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-order differential equation.

   (a) $(y - x)y' = y - x + 8$; $y = x + 4\sqrt{x+2}$

   (b) $y' = 2xy^2$; $y = 1/(4 - x^2)$

Solution: In order to check that the indicated function is an explicit solution of the given first-order differential equation, we need to substitute for y and y' and check that the equation holds.

(a) First find y':

$$\begin{aligned}\frac{d}{dx}(x + 4\sqrt{x+2}) &= \frac{d}{dx}(x) + \frac{d}{dx}(4(x+2)^{1/2}) \\ &= 1 + 4\left(\frac{1}{2}\right)(x+2)^{-1/2}\frac{d}{dx}(x+2) \\ &= 1 + \frac{2}{\sqrt{x+2}}\end{aligned}$$

Next substitute for y and y' on the left-hand side of the equation and check that it equals the right-hand side of the equation:

$$\begin{aligned}(y - x)y' &= (x + 4\sqrt{x+2} - x)\left(1 + \frac{2}{\sqrt{x+2}}\right) \\ &= 4\sqrt{x+2}\left(1 + \frac{2}{\sqrt{x+2}}\right) \\ &= 4\sqrt{x+2} + 8 \\ &= x + 4\sqrt{x+2} - x + 8 \\ &= y - x + 8\end{aligned}$$

(b) First find y':

$$\begin{aligned}\frac{d}{dx}(1/(4 - x^2)) &= \frac{d}{dx}((4 - x^2)^{-1}) \\ &= -(4 - x^2)^{-2}\frac{d}{dx}(4 - x^2) \\ &= \frac{2x}{(4 - x^2)^2}\end{aligned}$$

Next substitute for y and y' on the left-hand side of the equation and check that it equals the right-hand side of the equation:

$$\begin{aligned}
y' &= \frac{2x}{(4-x^2)^2} \\
&= 2x\left(\frac{1}{(4-x^2)^2}\right) \\
&= 2x\left(\frac{1}{(4-x^2)}\right)^2 \\
&= 2xy^2
\end{aligned}$$

5. For each of the following, solve the given differential equation by separation of variables.

   (a) $\frac{dy}{dx} = \sin(5x)$

   (b) $x\frac{dy}{dx} = 4y$

Solution: To solve a separable differential equation, first separate the equation into the form $p(y)dy = g(x)dx$. Then integrate both sides and solve for y (if needed).

(a) First separate the equation:

$$\begin{aligned} \frac{dy}{dx} &= \sin(5x) \\ dy &= \sin(5x)dx \end{aligned}$$

Next integrate both sides (use u-substitution in this case with $u = 5x$):

$$\begin{aligned} \int dy &= \int \sin(5x)dx \\ y &= \int \sin(u)\frac{1}{5}du \\ y &= \frac{1}{5}(-\cos(u)) + C \\ y &= -\frac{1}{5}\cos(5x) + C \end{aligned}$$

The equation is already solved for y, so this is our solution.

(b) First separate the equation:

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ \frac{dy}{y} &= \frac{4dx}{x} \end{aligned}$$

Next integrate both sides:

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{4dx}{x} \\ \ln|y| &= 4\ln|x| + C \end{aligned}$$

Last solve the equation for y (using logarithm properties in this case):

$$\boxed{\begin{aligned} \ln|y| &= 4\ln|x| + C \\ \ln|y| &= \ln(x^4) + C \\ e^{\ln|y|} &= e^{\ln(x^4)+C} \\ y &= Ax^4 \end{aligned}}$$

6. For each of the following, solve the given initial-value problem.

   (a) $\frac{dx}{dt} = 4(x^2+1)$, $x(\pi/4) = 1$

   (b) $x^2 \frac{dy}{dx} = y - yx$, $y(-1) = -1$

Solution: To solve an initial value problem, first find the solution to the differential equation. Then use the inital condition to solve for C.

(a) First solve the differential equation:

$$\begin{aligned} \frac{dx}{dt} &= 4(x^2+1) \\ \frac{dx}{x^2+1} &= 4dt \\ \int \frac{dx}{x^2+1} &= \int 4dt \\ \arctan(x) &= 4t + C \\ x &= \tan(4t + C) \end{aligned}$$

Next solve for C using the initial condition:

$$\begin{aligned} (1) &= \tan(4\left(\frac{\pi}{4}\right) + C) \\ 1 &= \tan(\pi + C) \\ \pi + C &= \frac{\pi}{4} \\ C &= -\frac{3\pi}{4} \end{aligned}$$

Thus the solution is $x = \tan(4t - \frac{3\pi}{4})$.

(b) First solve the differential equation:

$$\begin{aligned} x^2\frac{dy}{dx} &= y - yx \\ x^2\frac{dy}{dx} &= y(1-x) \\ \frac{dy}{y} &= \frac{1-x}{x^2}dx \\ \int \frac{dy}{y} &= \int \frac{1}{x^2}dx - \int \frac{1}{x}dx \\ \ln|y| &= -\frac{1}{x} - \ln|x| + C \end{aligned}$$

Next solve for C using the initial condition:

$$\begin{aligned} \ln|(-1)| &= -\frac{1}{(-1)} - \ln|(-1)| + C \\ 0 &= 1 - 0 + C \\ C &= -1 \end{aligned}$$

Thus the solution is $\ln|y| = -\frac{1}{x} - \ln|x| - 1$.