Name: Date: 04/13/2023

Math 20580 L.A. and D.E. Tutorial
Worksheet 12

Multiple-choice questions.

1. Figure 1 shows the direction field for the differential equation $\frac{dy}{dt} = f(y)$, where $f(y)$ is a polynomial of third degree.

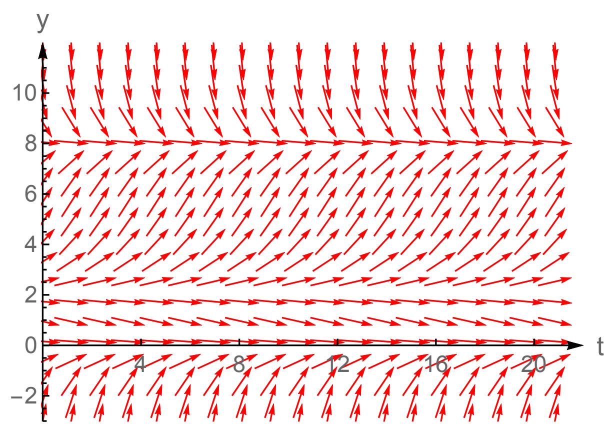

Figure 1

Which one of the following statements is FALSE?

A. The solution with initial value $y(0) = 9$ is decreasing and going to 8 as $t \to \infty$.
B. The **equilibrium** solutions of this differential equations are $y = 8$, $y = 2$ and $y = 0$.
C. $y = 8$ and $y = 0$ are asymptotically **stable** solutions.
D. $y = 2$ is an **unstable** solution.
E. The solution with initial value $y(0) = -2$ is increasing and becomes equal to 0 in **finite** time.

Solution: Looking at this direction field we see that all statements are true, except the statement: "The solution with initial value $y(0) = -2$ is increasing and becomes equal to 0 in finite time." In fact, the solution with initial value $y(0) = -2$ can not meet the equilibrium solution $y(t) = 0$ at some time $t_0 > 0$ because then we would have two solutions to the equation $\frac{dy}{dt} = f(y)$ with initial data $y(t_0) = 0$, contradicting the basic theorem about existence and uniqueness of solution when $f(y)$ is continuously differentiable. However, if we changed the statement to "The solution with initial value $y(0) = -2$ is increasing and goes to 0 as t goes to **infinity**," then that would be TRUE.

2. Which of the follow differential equations has the direction field shown in Figure 1 above?

   A. $\frac{dy}{dt} = (y-2)(y+2)(y-8)$

   B. $\frac{dy}{dt} = -y(1-\frac{y}{2})(1-\frac{y}{8})$

   C. $\frac{dy}{dt} = y(1-\frac{y}{2})(1-\frac{y}{8})$

   D. $\frac{dy}{dt} = -(2-y)^2(8-y)$

   E. $\frac{dy}{dt} = (2-y)^2(8-y)$

Solution: Looking at the direction field in Figure 1, we see that only the differential equation $\frac{dy}{dt} = -y(1-0.5y)(1-0.125y)$ has the equilibrium solutions $y = 0, 2, 8$ and the slopes of the arrows are consistent with the sign of the function $f(y) = -y(1-0.5y)(1-0.125y)$

3. Which of these ordinary differential equations are 2nd-order and non-linear?

   (1) $y'' + (\sin x)y' + (\tan x)y = e^x$

   (2) $y'' + (\sin x)y' + \tan(xy) = e^x$

   (3) $y' + \tan(xy) = e^x$

   (4) $y' + (\tan x)y = e^x$

   (5) $y''' + e^t y = 0$

   (6) $y'' + e^{ty} = 0$

   A. (2), (4) and (6)   B. (1), (3) and (4)   C. (2) and (6)   D. (5) and (6)

Solution:

(1) $y'' + (\sin x)y' + (\tan x)y = e^x$ is second-order and linear.

(2) ✓ $y'' + (\sin x)y' + \tan(xy) = e^x$ is second-order and nonlinear.

(3) $y' + \tan(xy) = e^x$ is first-order and nonlinear.

(4) $y' + (\tan x)y = e^x$ is first-order and linear.

(5) $y''' + e^t y = 0$ is third-order and linear.

(6) ✓ $y'' + e^{ty} = 0$ is second-order and nonlinear.

4. Find all the *stable* equilibrium solutions of the autonomous system

$$\frac{dy}{dt} = 6y - 5y^2 + y^3.$$

A. $y = 0$ B. $y = 0, y = 2$ C. $y = 2$ D. $y = 0, y = 2, y = 3$ E. $y = 3$

Solution:

The equilibria occur at solutions to $6y - 5y^2 + y^3 = 0$. Factor the LHS expression as $y(y-2)(y-3)$, so the solutions are $y = 0, 2, 3$.

For a stable equilibrium at $y = y_0$, we need $\frac{dy}{dt}$ to change sign from positive to negative as y crosses y_0. (Draw the diagram!)

Crossing 0, $\frac{dy}{dt} = y(y-2)(y-3)$ changes sign from negative to positive, so this equilibrium is unstable. The same thing happens at 3.

But when crossing 2, $y(y-2)(y-3)$ is positive when y is slightly less than 2, and $y(y-2)(y-3)$ is negative when y is slightly more than 2. So $y = 2$ is a stable equilibrium.

5. Determine the general solution of the differential equation

$$(x^2+1)\frac{dy}{dx} = y.$$

A. $y = \frac{x^3}{3} + x + c$
B. $y = c(x^2+1)$
C. $y = ce^{x^2+1}$
D. $y = ce^{\tan^{-1} x}$
E. $y = e^{\tan^{-1} x} + c$

Solution: We may solve for y by separation of variables.

$$\begin{aligned} \frac{1}{y}dy &= \frac{1}{x^2+1}dx \\ \int \frac{dy}{y} &= \int \frac{dx}{x^2+1} \\ \ln|y| &= \tan^{-1}(x) + c \\ |y| &= e^{\tan^{-1}(x)+c} \\ |y| &= e^{\tan^{-1}(x)}e^c \end{aligned}$$

We may rewrite this last expression as $y = ce^{\tan^{-1} x}$.

6. Determine an interval where the solution to the initial value problem is guaranteed to exist.

$$(t^2 - 4)y' = \sqrt{2-t}\ln(1+t), \quad y(0) = 0.$$

A. $-1 < t < 3$ B. $-1 < t$ C. $-1 < t < 2$ D. $t < 3$ E. $-2 < t$

Solution:

Rewrite this first-order, linear differential equation with initial condition as

$$y' = \frac{\sqrt{2-t}\ln(1+t)}{t^2-4}, \quad y(0) = 0.$$

The problem asks for the biggest open interval containing 0 over which the two functions of t are continuous. We need $t \le 2$ for the square root; $t > -1$ for the log function; and $t \neq 2, -2$ for the division by $t^2 - 4$. Hence $-1 < t < 2$.

Free response questions.

1. Consider the equation

$$y' = (y^3 - y)(9 - y^2)$$

with initial value $y(0) = 2$. Find $\lim_{t\to\infty} y(t)$.

Solution:

The constant solutions are $y = 0, y = \pm 1, y = \pm 3$. Since $y(0) = 2$, the solution lies between the lines $y = 1$ and $y = 3$. here $y' = (y^3 - y)(9 - y^2)$ is positive, since for instance at $y = 2$, the derivative is $(2^3 - 2)(9 - 2^2) = 6 \cdot 5 = 30$. Hence y is increasing and goes to 3 as t goes to infinity.

2. Let $y = \phi(x)$ be a solution to $\dfrac{dy}{dx} = \dfrac{\cos^2(y)}{x^2}$ that satisfies $\phi(1) = 0$. What is the interval of definition of $\phi(x)$? Find $\phi(2)$.

Solution:

This is a first-order, separable, non-linear differential equation. The interval of definition of $y = \phi(x)$ is the largest interval containing 1 on which $\frac{1}{x^2}$ is defined. This is $(0, \infty)$.

To find $\phi(2)$, rewrite the equation as $\dfrac{dy}{\cos^2 y} = \dfrac{dx}{x^2}$, and integrate both sides:

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{x^2} \implies \tan(y) = -\frac{1}{x} + C.$$

Since $\phi(x)$ is a solution to the given differential equation, it satisfies $\tan(\phi) = -\frac{1}{x}+C$, so $\phi(x) = \arctan\left(-\frac{1}{x} + C\right)$.

We need to find $\phi(2)$, but before that we need to know C, which we will find using the information $\phi(1) = 0$.

$$\phi(1) = \arctan(-1 + C) = 0 \implies -1 + C = 0 \implies C = 1.$$

Therefore $\phi(x) = \arctan\left(-\frac{1}{x} + 1\right) \implies \phi(2) = \arctan\left(-\frac{1}{2} + 1\right) = \arctan\left(\frac{1}{2}\right)$.

3. Which of the following are first-order linear differential equations? Check **all** that apply:

- ☐ $y' = \dfrac{M(x)}{N(y)}$.
- ☐ $y'' + P(x)y = Q(x)$.
- ☐ $y' + P(x)y = Q(x)$.
- ☐ $P(x)y' + y = Q(x)y^2$.
- ☐ $P(x)y' + Q(x)y = R(x)$.
- ☐ $y' = P(x) + Q(x)y$.

Write the formula for the integrating factor for each linear equation you found above.

Solution:

1. $y' = \dfrac{M(x)}{N(y)}$ — first-order, not necessarily linear (depending on what $N(y)$ is), separable.
2. $y'' + P(x)y = Q(x)$ — second-order, linear.
3. ✓ $y' + P(x)y = Q(x)$ — first-order, linear.
   The integrating factor is $\mu(x) = e^{\int P(x)dx}$.
4. $P(x)y' + y = Q(x)y^2$ — first-order, non-linear.
5. ✓ $P(x)y' + Q(x)y = R(x)$ — first-order, linear.
   We can rearrange this differential equation into $y' + \frac{Q(x)}{P(x)}y = \frac{R(x)}{P(x)}$, so the integrating factor is $\mu(x) = e^{\int \frac{Q(x)}{P(x)}dx}$.

6. ✓ $y' = P(x) + Q(x)y$ — first-order, linear.

   We can rearrange this differential equation into $y' - Q(x)y = P(x)$, so the integrating factor is $\mu(x) = e^{\int -Q(x)dx}$.

4. Solve the differential equation $y' = x^3y + e^{x^4/4}\sin x$ with $y(0) = 1$.

Solution:

This is a first-order, linear differential equation.

Rewrite it as $y' - x^3y = e^{x^4/4}\sin x$, and find the integrating factor:

$$\mu(x) = e^{\int -x^3\,dx} = e^{-x^4/4}.$$

Multiply both sides of the differential equation by the integrating factor to get

$$e^{-x^4/4}y' - e^{-x^4/4}xy = e^{-x^4/4}e^{x^4/4}\sin x \implies \left(e^{-x^4/4}y\right)' = \sin x.$$

Continue using the integrating factor:

$$e^{-x^4/4}y = \int \sin x\,dx \implies e^{-x^4/4}y = -\cos x + C \implies y = -e^{x^4/4}\cos x + Ce^{x^4/4}.$$

Find C:

$$y(0) = 1 \implies -1 + C = 1 \implies C = 2.$$

Therefore the solution of the differential equation is

$$y = -e^{x^4/4}\cos x + 2e^{x^4/4}.$$

5. Consider the differential equation

$$(y - \cos(x)) + (x + \sin(y))\frac{dy}{dx} = 0.$$

(a) Is the above differential equation exact?

Solution: $M = y - \cos(x) \implies M_y = 1.$

$N = x + \sin(y) \implies N_x = 1.$

$M_y = N_x \implies$ exact.

(b) Find the solution of the above differential equation.

Solution: *Answer:* $yxy - \sin(x) - \cos(y) = c$

We want to find $\psi(x, y)$ such that $\psi_x = y - \cos(x)$ and $\psi_y = x + \sin(y)$.

The first condition $\psi_x = y - \cos(x)$ implies $\psi(x, y) = \int y - \cos(x)\, dx = xy - \sin(x) + h(y)$. This also gives $\psi_y = x + h'(y)$, which we will now use to find $h(y)$.

The second condition $\psi_y = x + \sin(y)$ implies $\psi_y = x + h'(y) = x + \sin(y) \implies h'(y) = \sin(y) \implies h(y) = -\cos(y)$.

Therefore $\psi(x, y) = xy - \sin(x) - \cos(y)$, and the solution to the differential equation is $xy - \sin(x) - \cos(y) = c$ for any constant c.

6. Consider the differential equation

$$2xy^2 + (4x^2y + 3)y' = 0$$

Is it exact? If not, does it have an integrating factor that is a function of x or y alone? If so, solve the differential equation with initial value $y(1) = -1$. (Express your solution as $y =$ some expression in x.)

Solution:

The differential equation is **not** exact:

$M = 2xy^2 \implies M_y = 4xy$, while $N = 4x^2y + 3 \implies N_x = 8xy$. Since $M_y \neq N_x$, the differential equation is not exact. However, we can find an integrating factor:

$M_y - N_x = -4xy \implies \dfrac{M_y - N_x}{M} = \frac{-2}{y}$ is a function of y alone.

So we have an integrating factor $\mu(y)$ which is a function of y alone, and which satisfies $\dfrac{d\mu}{dy} = -\dfrac{M_y - N_x}{M}\mu = \frac{2}{y}\mu$.

We can solve this separable first-order linear differential equation to find a $\mu(y)$:

$$\int \frac{d\mu}{\mu} = \int \frac{2}{y} dy \implies \ln|\mu| = 2\ln|y| \implies \mu(y) = y^2 \text{ works.}$$

Now multiply through by $\mu(y) = y^2$ to get

$$2xy^4 + (4x^2y^3 + 3y^2)y' = 0$$

which is an exact equation. (You may check this.) So far we have turned our original "almost exact" equation into this "exact" equation. Now we just need to solve this exact equation. We need to find a function $\psi(x, y)$ such that $\psi_x = 2xy^4$ and $\psi_y = 4x^2y^3 + 3y^2$:

1. The first condition $\psi_x = 2xy^4$ gives $\psi(x, y) = \int 2xy^4 dx = x^2y^4 + h(y)$.
2. Take the partial derivative of $\psi(x, y) = x^2y^4 + h(y)$ w.r.t. y to get $\psi_y = 4x^2y^3 + h'(y)$, and compare with the second condition $\psi_y = 4x^2y^3 + 3y^2$ to get that $h'(y) = 3y^2$. Hence $h(y) = y^3$ works.

Now we found $\psi(x, y) = x^2y^4 + y^3$, so a general solution of the given differential equation looks like $x^2y^4 + y^3 = c$ for any constant c.

If we want to find a specific solution that takes into account the initial condition $y(1) = -1$, then we have to use that condition to find the constant c:

$$y(1) = -1 \implies c = 0.$$

Finally we need to single out one of many possible solutions that are described by this equation $x^2y^4 + y^3 = 0$. We begin by factoring:

$$0 = x^2y^4 + y^3 = y^3(x^2y + 1)$$

As $y(1) \neq 0$, the solution we want is $x^2y + 1 = 0$. In other words, $y = \frac{-1}{x^2}$.