Name: SOLUTIONS Date: 04/20/2023

M20580 L.A. and D.E. Tutorial
Worksheet 13

1. Which of the following is true about the differential equation

$$(3y^2 - 4x(y^3+1))dx + xy(2-3xy)dy = 0.$$

   (a) It is exact.
   (b) It is homogeneous.
   (c) It has an integrating factor that is a function of x alone.
   (d) It has an integrating factor that is a function of y alone.
   (e) None of the above.

Solution: The statement in (a) is false. If $M(x,y) = 3y^2 - 4x(y^3+1)$ and $N(x,y) = xy(2-3xy)$, then $M_y = 6y - 12xy^2$ and $N_x = 2y - 6xy^2$. Since $M_y \neq N_x$, the differential equation is not exact.

The statement in (b) is false. The differential equation is not homogeneous since it cannot be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

The statement in (c) is true. Notice that

$$\frac{M_y - N_x}{N} = \frac{4y - 6xy^2}{xy(2-3xy)} = \frac{2}{x}$$

is a function of x alone, so the differential equation has an integrating factor that is a function of x alone and it is given by

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}.$$

The statement in (d) is false. Notice that

$$\frac{N_x - M_y}{M} = \frac{-4y + 6xy^2}{3y^2 - 4x(y^3+1)}$$

does not seem to simplify.

2. Solve the differential equation

$$(3y^2 - 4x(y^3+1))dx + xy(2-3xy)dy = 0.$$

Solution: By (c) in the previous problem, this differential equation is not exact, but it has an integrating factor that is a function of x alone, and is given by

$$\mu(x) = e^{\int \frac{2}{x} dx} = x^2.$$

Multiplying both sides of the differential equation by $\mu(x)$, we obtain

$$x^2(3y^2 - 4x(y^3 + 1))dx + x^3y(2 - 3xy)dy = 0.$$

and you can check that it is exact. Hence, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = x^2(3y^2 - 4x(y^3 + 1)) \quad \text{and} \quad \frac{\partial f}{\partial y} = x^3y(2 - 3xy).$$

From the first equation, $f(x, y) = x^3y^2 - x^4y^3 - x^4 + g(y)$. Taking the partial derivative of this expression with respect to y and setting it equal to $x^3y(2-3xy)$ gives $g'(y) = 0$, so $g(y) = c$ for some arbitrary constant c. Hence, $f(x, y) = x^3y^2 - x^4y^3 - x^4 + c$. This implies that the solution of the differential equation is $x^3y^2 - x^4y^3 - x^4 + c = c'$ for some arbitrary constant c'.

3. For each one of the following differential equations, consider the given set of functions of x. You can show that the given set is a set of solutions of the corresponding differential equation. Determine whether the set is linearly independent.

   (a) $y'' = 0$ and $\{0, x\}$.

   (b) $y''' = 0$ and $\{1, x, x^2\}$.

   (c) $xy'' - y' - 4x^3y = 0$ and $\{e^{x^2}, e^{-x^2}\}$.

   (d) $y'' - 3y' + 2y = 0$ and $\{e^x, e^{2x}\}$.

Solution: Recall that a set of solutions of a differential equation is linearly independent if the Wronskian of those functions is not identically zero, so we compute the Wronskian W of each set.

(a) $W = 0$.

(b) $W = 2$.

(c) $W = -4x$.

(d) $W = e^{3x}$.

The set in (a) is linearly dependent since its Wronskian is 0. Another way to conclude that the set is linearly dependent is using linear algebra: any set containing the zero vector is linearly dependent. The sets that are linearly independent are the ones in (b), (c) and (d).

4. For each of the following differential equations, first check if it is linear and then solve it using an appropriate method.
   (a) $\frac{dy}{dt} = ty^2 \cos t$, $y(0) = 1$
   (b) $t\frac{dy}{dt} = t^2 + y$, $y(1) = -2$, $t > 0$,
   (c) $y' = 2ty + 3t^2e^{t^2}$
   (d) $t^2\frac{dy}{dt} + ty = 1$, $t > 0$.

Solution:

(a) This differential equation is not linear. Using the method of separation of variables, we have:

$$\begin{aligned}
\frac{dy}{dt} = ty^2\cos t &\implies \frac{1}{y^2}dy = t\cos t\, dt \\
&\implies \int y^{-2}dy = \int t\cos t\, dt \\
&\qquad\qquad\qquad \text{Integration by parts:} \\
&\qquad\qquad\qquad u = t \implies du = dt \\
&\qquad\qquad\qquad dv = \cos t\, dt \implies v = \sin t \\
&\implies \frac{y^{-1}}{-1} = t\sin t - \int \sin t\, dt \\
&\implies -y^{-1} = t\sin t - (-\cos t) + C \\
&\implies -y^{-1} = t\sin t + \cos t + C
\end{aligned}$$

Since $y(0) = 1$, we have

$$-1^{-1} = 0\sin 0 + \cos 0 + C \implies C = -1^{-1} - \cos 0 = -1 - 1 = -2$$

Therefore,

$$\begin{aligned}
-y^{-1} = t\sin t + \cos t - 2 &\implies y^{-1} = 2 - t\sin t - \cos t \\
&\implies y = \frac{1}{2 - t\sin t - \cos t}.
\end{aligned}$$

(b) This differential equation is linear. We have

$$t\frac{dy}{dt} = t^2 + y \implies t\frac{dy}{dt} - y = t^2 \implies \frac{dy}{dt} - t^{-1} \cdot y = t.$$

Multiplying both sides of the equation by the following integrating factor

$$\begin{aligned} \mu &= e^{\int -t^{-1} dt} = e^{-\ln|t|} = e^{-\ln t} && (|t| = t \text{ as } t > 0) \\ &= t^{-1}, && (\text{Note: } e^{-\ln x} \text{ is NOT } -x) \end{aligned}$$

we then have

$$\begin{aligned} t^{-1}\frac{dy}{dt} - t^{-2}y = 1 &\implies (t^{-1}y)' = 1 \\ &\implies t^{-1}y = \int 1 \cdot dt \\ &\implies t^{-1}y = t + C \\ &\implies y = t^2 + Ct. \end{aligned}$$

Since $y(1) = -2$, we have

$$-2 = 1^2 + C \cdot 1 \implies C = -2 - 1 = -3.$$

Therefore, $y = t^2 - 3t$.

(c) This differential equation is linear. We have

$$y' = 2ty + 3t^2e^{t^2} \implies y' - 2t \cdot y = 3t^2e^{t^2}.$$

We multiply both sides of the equation by the following integrating factor

$$\mu = e^{\int(-2t)dt} = e^{-t^2}$$

to obtain

$$\begin{aligned} y'e^{-t^2} - 2te^{-t^2}y = 3t^2 &\implies (ye^{-t^2})' = 3t^2 \\ &\implies ye^{-t^2} = \int 3t^2 dt \\ &\implies ye^{-t^2} = t^3 + C. \end{aligned}$$

Therefore, $y = e^{t^2}(t^3 + C)$, where C is an arbitrary constant.

(d) This differential equation is linear. We have

$$t^2\frac{dy}{dt} + ty = 1 \implies \frac{dy}{dt} + t^{-1} \cdot y = t^{-2}.$$

We multiply both sides of the equation by the following integrating factor

$$\mu = e^{\int t^{-1} dt} = e^{\ln|t|} = |t| = t. \qquad (|t| = t \text{ for } t > 0)$$

to obtain

$$\begin{aligned} t\frac{dy}{dt} + y = t^{-1} &\implies (ty)' = t^{-1} \\ &\implies ty = \int t^{-1} dt \\ &\implies ty = \ln|t| + C \\ &\implies ty = \ln t + C. \qquad (|t| = t \text{ for } t > 0) \end{aligned}$$

Therefore, $y = t^{-1} \ln t + Ct^{-1}$, where C is an arbitrary constant.

5. A tank originally has 100 liters of a brine with a concentration of 0.05 grams of salt per liter. Brine with concentration of 0.02 grams of salt per liter is pumped into the tank at a rate of 5 liters per second. The mixture is kept stirred and is pumped out at a rate of 4 liters per second. Find the amount of salt in the tank as a function of time.

Solution: Let $y(t)$ be the amount of salt in the tank at t minutes. Then

$$\frac{dy}{dt} = \text{rate of incoming salt} - \text{rate of outgoing salt},$$

where

$$\begin{aligned} \text{rate of incoming salt} &= (\text{rate of incoming volume of brine})(\text{incoming density}) \\ &= 5(0.02) = 0.1 \end{aligned}$$

and

$$\begin{aligned} \text{rate of outgoing salt} &= (\text{rate of outgoing volume of brine})(\text{outgoing density}) \\ &= 4\left(\frac{y(t)}{100 + (5-4)t}\right) = \frac{4y(t)}{100 + t}. \end{aligned}$$

From here, we obtain the first order linear differential equation

$$\frac{dy}{dt} = 0.1 - \frac{4}{100 + t} y.$$

Solving this differential equation, we obtain

$$y(t) = 0.02(100 + t) + C(100 + t)^{-4}.$$

(Note: Don't be tempted to expand $(100+t)^4$) With the initial condition $y(0) = 100(0.05) = 5$, we have $C = 3(100^4) = 3(10^8)$, so

$$y(t) = 0.02(100+t) + \frac{3(10^8)}{(100+t)^4} \quad \text{grams.}$$

6. Suppose that a population of rabbits on a meadow initially has size 10000 and natural birth rate 10% per year. A pack of 20 wolves just migrated in recently and, as a result, each wolf hunts for two rabbits each month for food. Suppose that the population of wolves grows by 2 wolves/year. Set up a differential equation describing the population of rabbits over time. Will the population of rabbits eventually flourish, stay the same, or go extinct?

Solution: Let $y(t)$ denotes the population of rabbits after t years. Then

$$\frac{dy}{dt} = \text{natural growth rate} - \text{rate of decrease}$$

In this case, the decreasing rate of rabbits are determined by the hunting rate of the wolves. After t years, the wolf population is $20 + 2t$, so the rabbit population decreases by a rate of $2(12)(20 + 2t) = 48(t + 10)$. Hence, we have

$$\frac{dy}{dt} = 0.1y - 48(t+10).$$

This is a first order linear differential equation. Solving for y, we obtain

$$y(t) = 48(10t + 200) + Ce^{0.1t}.$$

Using the initial condition $y(0) = 5000$, we have $C = 400$. Hence,

$$y(t) = 480t + 9600 + 400e^{0.1t}.$$

Since $y'(t) = 480 + 40e^{0.1t} > 0$, the population of rabbits will keep increasing so it will eventually flourishes.

7. Consider the differential equation

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0$$

and assume that $y_1(t) = t$ is a solution. Use the reduction of order method to find a second linearly independent solution.

Solution: First, set $y = u(t)t$. Then

$$y' = u + tu' \quad \text{and} \quad y'' = tu'' + 2u'.$$

Replacing y, y' and y'' in the differential equation, and collecting terms, we obtain

$$t^2(tu'' + 2u') + 2t(u + tu') - 2(tu) = t^3u'' + 4t^2u' = 0.$$

Solving for u', we find that $u' = ct^{-4}$. It follows that $u = \frac{1}{-3}ct^{-3} + k$, and hence

$$y = ut = \frac{1}{-3}ct^{-2} + kt$$

for arbitrary constants c and k. Notice that the second term on the right side of the previous equation is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have $y_2(t) = t^{-2}$.

8. Consider the differential equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

and assume that $y_1(t) = t^{-1}$ is a solution. Use the reduction of order method to find a second linearly independent solution.

Solution: First, set $y = u(t)t^{-1}$. Then

$$y' = u't^{-1} - ut^{-2} \quad \text{and} \quad y'' = u''t^{-1} - 2u't^{-2} + 2ut^{-3}.$$

Replacing y, y' and y'' in the differential equation, and collecting terms, we obtain

$$\begin{aligned} 2t^2(u''t^{-1} - 2u't^{-2} + 2ut^{-3}) + 3t(u't^{-1} - ut^{-2}) - (ut^{-1}) \\ &= 2tu'' + (-4+3)u' + (4t^{-1} - 3t^{-1} - t^{-1})u \\ &= 2tu'' - u' \\ &= 0. \end{aligned}$$

Solving for u', we find that $u' = ct^{1/2}$. It follows that $u = \frac{2}{3}ct^{3/2} + k$, and hence

$$y = ut^{-1} = \frac{2}{3}ct^{1/2} + kt^{-1}$$

for arbitrary constants c and k. Notice that the second term on the right side of the previous equation is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have $y_2(t) = t^{1/2}$.