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M20580 L.A. and D.E. Tutorial
Worksheet 14

1. Find a homogeneous linear differential equation with constant coefficients whose general solution is

   (a) $y = c_1 e^{x} + c_2 e^{5x}$

   (b) $y = c_1 + c_2 e^{2x}$

   (c) $y = c_1 e^{-4x} + c_2 e^{-3x}$

   (d) $y = c_1 \cos 3x + c_2 \sin 3x$

   (e) $y = c_1 e^{10x} + c_2 x e^{10x}$

   (f) $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

Solution:

(a) We have $(r-1)(r-5) = r^2 - 6r + 5$. So, the differential equation is $y'' - 6y' + 5y = 0$.

(b) We have $r(r-2) = r^2 - 2r$. So, the differential equation is $y'' - 2y' = 0$.

(c) We have $(r+4)(r+3) = r^2 + 7r + 12$. So, the differential equation is $y'' + 7y' + 12y = 0$.

(d) We have $(r-3i)(r+3i) = r^2 + 9$. So, the differential equation is $y'' + 9y = 0$.

(e) We have $(r-10)^2 = r^2 - 20r + 100$. So, the differential equation is $y'' - 20y' + 100y = 0$.

(f) We have $(r-(-1+i))(r-(-1-i)) = r^2 + 2r + 2$. So, the differential equation is $y'' + 2y' + 2y = 0$.

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2. Use reduction of order to find a second solution to: $y'' - 4y' + 4y = 0$; $y_1 = e^{2x}$.

Solution: Define $y = u(x)e^{2x}$ so $y' = 2ue^{2x} + u'e^{2x}$, $y'' = e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u$, and $y'' - 4y' + 4y = e^{2x}u'' = 0$. Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution is $y_2 = xe^{2x}$.

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3. (i) Solve the initial-value problem: $4y'' - 4y' - 3y = 0$, $y(0) = 1$, $y'(0) = 5$.

(ii) Using method of indeterminate coefficients find a particular solution of $4y'' - 4y' - 3y = 3x - 1$. (Hint: try linear function $Ax + B$).

(iii) Find the general solution of the $4y'' - 4y' - 3y = 3x - 1$.

Solution: (i) The auxiliary equation is $4r^2 - 4r - 3 = 0$ and the roots are $-1/2$ and $3/2$. So, the general solution is $y = c_1 e^{-x/2} + c_2 e^{3x/2}$. Since $y(0) = 1$ and $y'(0) = 5$, then $c_1 + c_2 = 1$ and $(-1/2)c_1 + (3/2)c_2 = 5$. Solving the linear system gives $c_1 = -7/4$ and $c_2 = 11/4$. So, the solution to the IVP is $y = (-7/4)e^{-x/2} + (11/4)e^{3x/2}$.

(ii) Plug in the function Ax+B in the equation and search for coefficients A and B: $-4A - 3(Ax + B) = 3x - 1$. Thus, $-4A - 3B = -1$ and $-3A = -3$. Thiwc implies that $A = 1$ and $B = \frac{5}{3}$. Thus, the particular solution is $-x + \frac{5}{3}$.

(iii) Thus, the general solution is $y = -x + 5 + c_1 e^{-x/2} + c_2 e^{3x/2}$.

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4. (i) Solve the initial-value problem: $4y'' - 4y' + y = 0$, $y(0) = 1$, $y'(0) = 2$.

   (ii) Using method of indeterminate coefficients find a particular solution of $4y'' - 4y' + y = -9e^{2x}$. (Hint: try linear function Ae^{2x}).

   (iii) Find the general solution of the $4y'' - 4y' + y = -9e^{2x}$.

Solution: (i) The auxiliary equation is $4r^2 - 4r + 1 = 0$ and the root is 1/2 of multiplicity 2. So, the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Since $y(0) = 1$ and $y'(0) = 2$, then $c_1 + c_2 = 1$ and $(1/2)c_1 + c_2 = 2$. Solving the linear system gives $c_1 = -2$ and $c_2 = 3$. So, the solution to the IVP is $y = -2e^{-x/2} + 3e^{3x/2}$.

(ii) Plug in the function Ae^{2x} in the equation and search for coefficient A: $16A - 8A + A = -9$ Thus, the particular solution is $-e^{2x}$.

(iii) Thus, the general solution is $y = -e^{2x} + -2e^{-x/2} + 3e^{3x/2}$.

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5. (i) Solve the boundary value problem $y'' - 2y' + 2y = 0$, $y(0) = 1$, $y(\pi) = 1$.

   (ii) Using method of indeterminate coefficients find a particular solution of $y'' - 2y' + 2y = -\cos(x)$. (Hint: try linear function $A\sin(x) + B\cos(x)$).

   (iii) Find the general solution of the $y'' - 2y' + 2y = -\cos(x)$.

Solution: The auxiliary equation is $r^2 - 2r + 2 = 0$ and the roots are $1 \pm i$. So, the general solution is $y = e^x(c_1 \cos x + c_2 \sin x)$. Since $y(0) = 1$ and $y(\pi) = 1$, then we find that $c_1 = 1$ but we also find that $-e^\pi = 1$ which is a contradiction. So, the BVP has no solution.

(ii) Plug in the function $A\sin(x)+B\cos(x)$ in the equation and search for coefficients A and B: $A + 2B = 0$ and $B - 2A = -1$ Thus, the particular solution is $\frac{2}{5}\sin(x) - \frac{1}{5}\cos(x)$.

(iii) Thus, the general solution is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{2}{5}\sin(x) - \frac{1}{5}\cos(x)$.

6. Suppose t and $\sin t$ are two solutions to the homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Use variation of parameters to find a solution to the non-homogeneous differential equation

$$y'' + p(t)y' + q(t)y = t\cos t - \sin t.$$

Hint: method of variation of parameters suggests to look for solutions of the form $y_1u_1(t) + y_2u_2(t)$, where y_1 and y_2 is FSS. These functions could be found from the following:

$$u_1(t) = \int -\frac{g(t)y_2(t)}{W(y_1, y_2)}\, dt$$

and

$$u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\, dt$$

Solution:

Here $y_1 = t, y_2 = \sin t$ and $g = t\cos t - \sin t$.

The Wronskian is $W(y_1, y_2) = \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} = t\cos t - \sin t = g(t)$.

Hence

$$u_1(t) = \int -\frac{g(t)y_2(t)}{W(y_1, y_2)}\, dt = \int -\sin t\, dt = \cos t,$$

and

$$u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\, dt = \int t\, dt = \frac{1}{2}t^2.$$

So this method gives the particular solution

$$y_p = u_1y_1 + u_2y_2 = t\cos t + \frac{1}{2}t^2 \sin t.$$

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7. Solve the differential equation $xy'' + 2y' + xy = 0$. (Hint: consider $u = xy$)

Solution: As is, the equation can be classified as second order linear homogeneous with nonconstant coefficients, but it can be rewritten as $(xy)'' + (xy) = 0$. Solving, we find $xy = c_1 \cos x + c_2 \sin x$, and hence

$$y = c_1 \frac{\cos x}{x} + c_2 \frac{\sin x}{x}.$$