Name: SOLUTIONS Date: 01/26/2023

M20580 L.A. and D.E. Tutorial
Worksheet 2

1. Determine whether the vector $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If yes, find scalars a_1, a_2, a_3 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

Solution: To solve $a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3 = \mathbf{w}$, row reduce the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -3 & 1 & -6 & -5 \\ 0 & 1 & 8 & 6 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 9 & 1 \\ 0 & 1 & 8 & 6 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

By the third row, we see $a_3 = -5$. Similarly, we obtain $a_2+9a_3 = 1$ and $a_1+5a_3 = 2$. Thus, $(a_1, a_2, a_3) = (27, 46, -5)$.

2. Find the inverses of the following matrices if it exists

(a) $\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -6 \\ 6 & -9 \end{bmatrix}$

Solution: We recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad - bc \neq 0$, then the inverse of A is given by $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, then A does not have an inverse (not invertible).

(a) $2 \times 4 - 7 \times 3 = -13 \neq 0$, so the inverse is

$$\frac{1}{-13}\begin{bmatrix} 2 & -3 \\ -7 & 4 \end{bmatrix}$$

(b) $4 \times (-9) - 6 \times (-6) = 0$ so the matrix is not invertible.

3. (a) Let $A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$.

Construct a 2×2 matrix B such that AB is the zero matrix. Use two different **nonzero** columns for B.

Solution: We note that if we write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{b}_1, \mathbf{b}_2$ are two column vectors in $\mathbb{R}^2$, then $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$ (See, e.g., textbook Theorem 3.3 and its proof). Hence, AB is a zero matrix if and only if $A\mathbf{b}_1 = A\mathbf{b}_2 = \mathbf{0}$, so we need to find two (nonzero) solutions to the system $A\mathbf{x} = \mathbf{0}$.

Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, so

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x_1 - 3x_2 & = 0 \\ 0x_1 + 0x_2 & = 0 \quad \text{(trivial)} \end{cases} \iff x_1 = 3x_2.$$

We can choose $x_2 = 1$ and then $x_2 = -1$, which gives $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, so one choice for a matrix B that satisfies $AB = 0$ is $B = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Find the conditions on a, b, c, and d such that A commutes with both B and C, that is, $AB = BA$ and $AC = CA$.

Solution: We can work out to see that

$$AB = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}, \qquad BA = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix},$$

so by comparing entry-by-entry, we see that $AB = BA$ if and only if $b = 0$ and $c = 0$.

Likewise, we have

$$AC = \begin{bmatrix} a & -2a + b \\ c & -2c + d \end{bmatrix}, \qquad CA = \begin{bmatrix} a - 2c & b - 2d \\ c & d \end{bmatrix},$$

so that $AC = CA$ if and only if $a = a - 2c$ and $-2a + b = b - 2d$ and $-2c + d = d$. Solving these three conditions simultaneously, we obtain $c = 0$ and $a = d$.

Hence, A commutes with both B and C if $b = c = 0$ and $a = d$, where a can be any number.

4. (a) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation. If $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, T(\mathbf{w}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$, where $\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} + \mathbf{w}$.

Solution: We recall two properties of a linear transformation T: for any two vectors of appropriate size $\mathbf{u}, \mathbf{v}$ and any scalar c, we have

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Now, we continually use these two properties to compute $T(\mathbf{x})$. We have

$$\begin{aligned} T(\mathbf{x}) &= T(\mathbf{2u} + 3\mathbf{v} + \mathbf{w}) = T(2\mathbf{u}) + T(3\mathbf{v}) + T(\mathbf{w}) && \text{(property (1))} \\ &= 2T(\mathbf{u}) + 3T(\mathbf{v}) + T(\mathbf{w}) && \text{(property (2))} \\ T(\mathbf{x}) &= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}. \end{aligned}$$

(b) Continuing part (a), if we know $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x}$ in $\mathbb{R}^3$.

Solution: If we write $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for any vector in $\mathbb{R}^3$, we have

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

so using the same argument as in (b), we have

$$\begin{aligned} T(\mathbf{x}) &= aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) \\ &= a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} a + 3b + 4c \\ 2a + b + 2c \end{bmatrix} \\ T(\mathbf{x}) &= \begin{bmatrix} \langle 1, 3, 4 \rangle \cdot \langle a, b, c \rangle \\ \langle 2, 1, 2 \rangle \cdot \langle a, b, c \rangle \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}\mathbf{x}. \end{aligned}$$

Therefore, $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$.

5. For each of the following linear transformation $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, find the standard matrix for T, i.e., find a 2×2 matrix A such that $T\mathbf{x} = A\mathbf{x}$.

   (a)
$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \forall \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

   (b)
$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 3x_2 \end{bmatrix}, \forall \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

Solution:

(a)
$$A = [T]_{std}^{std} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b)
$$A = [T]_{std}^{std} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 - 2 \times 0 \\ 3 \times 0 \end{bmatrix} & \begin{bmatrix} 0 - 2 \times 1 \\ 3 \times 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}.$$

6. * Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \tag{1}$$

(a) Explain why B is invertible.

(b) Solve (1) for X. If you need to invert a matrix, explain why that matrix is invertible.

Solution:

(a) Since $A - AX$ is invertible, multiplying to the left of both sides of (1) by $(A - AX)$, we have

$$I_n = [(A - AX)X^{-1}]B$$

so if we let $Y = (A - AX)X^{-1}$, we have $YB = I_n$. By a Theorem in class (or see, e.g, Poole's book Theorem 3.13), B is invertible whose inverse is Y.

(b) Now since B is invertible, multiplying to the left of both sides of (1) by X, we have

$$\begin{aligned} X(A - AX)^{-1} &= (XX^{-1})B \\ X(A - AX)^{-1} &= I_nB \\ X(A - AX)^{-1} &= B \end{aligned}$$

Since $A - AX$ is invertible, multiplying to the right of both sides by $A - AX$, we have

$$\begin{aligned} XI_n &= B(A - AX) \\ X &= BA - BAX \end{aligned}$$

Adding both sides by BAX, we have

$$\begin{aligned} X + BAX &= BA \\ I_nX + BAX &= BA \\ (I_n + BA)X &= BA \end{aligned}$$

Multiplying to the right of both sides by X^{-1}, we have

$$I_n + BA = BAX^{-1}$$

Multiplying to the left of both sides by B^{-1} (which exists by part (a)), we have

$$B^{-1}(I_n + BA) = AX^{-1}$$

Finally, multiplying to the left of both sides by A^{-1}, we have

$$A^{-1}B^{-1}(I_n + BA) = X^{-1}.$$

Therefore, $X^{-1} = A^{-1}B^{-1}(I_n + BA)$, so that

$$\begin{aligned} X &= [A^{-1}B^{-1}(I_n + BA)]^{-1} = (I_n + BA)^{-1}(B^{-1})^{-1}(A^{-1})^{-1} \\ X &= (I_n + BA)^{-1}BA. \end{aligned}$$