Name: SOLUTIONS Date: 02/02/2023

M20580 L.A. and D.E. Tutorial
Worksheet 3

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformations defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+3y \\ 2x+y \\ x-y \end{bmatrix},$$

respectively. Find the standard matrix for $S \circ T$, i.e., the matrix A (of appropriate dimensions) such that $(S \circ T)(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. Determine whether it is possible to find the standard matrix for $T \circ S$.

Solution: By Theorem 3.32 in Poole's book, the standard matrix $M_{S\circ T}$ of the linear transformation $S \circ T$ satisfies $M_{S\circ T} = M_S M_T$, where M_T and M_S are the standard matrices of the linear transformations T and S, respectively. Notice that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so the standard matrix of T is given by $M_T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Similarly, notice that

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix},$$

so the standard matrix of S is given by $M_S = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore, the standard matrix $M_{S\circ T}$ of $S \circ T$ is given by

$$M_{S\circ T} = M_S M_T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Finally, it is not possible to find the standard matrix for $T \circ S$ because this composition does not make sense. The linear transformation S returns a vector in $\mathbb{R}^3$, while the argument of the linear transformation T should be a vector in $\mathbb{R}^2$.

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2. Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

Solution: If three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ are linearly dependent, then at least one of them must be in the span of the others. Since we want any two of these vectors to be linearly independent, let us start with two linearly independent vectors. Two such vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These two vectors span the plane $z = 0$ in $\mathbb{R}^3$. Now, we want to choose one more vector, $\mathbf{v}_3$, such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but any two of these vectors are linearly independent. Equivalently, we want to choose $\mathbf{v}_3$ such that $\mathbf{v}_3$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$, but $\mathbf{v}_3$ is not in the span of $\{\mathbf{v}_1\}$ and not in the span of $\{\mathbf{v}_2\}$. One such vector is

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ satisfies the required conditions. We emphasize that this solution is not unique.

3. Determine if the following vectors are linearly independent.

(a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

(b) $\begin{bmatrix}7\\10\\3\end{bmatrix}, \begin{bmatrix}5\\7\\2\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}$

Solution:

(a) The matrix

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

is in echelon form, so the three vectors are linearly independent.

(b) The matrix

$$\begin{bmatrix}7 & 5 & 2\\10 & 7 & 3\\3 & 2 & 1\end{bmatrix}$$

can be reduced to obtain the matrix

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}.$$

We conclude that the three vectors are linearly dependent.

4. Show that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \tag{1}$$

form a basis for $\mathbb{R}^3$.

Solution: The matrix $\begin{bmatrix} 2 & 1 & -1 \\ -3 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent and span all $\mathbb{R}^3$. Therefore, they form a basis for $\mathbb{R}^3$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2x + z \\ 5x \\ x - 3z \end{bmatrix}. \tag{2}$$

   (a) Find the standard matrix of T.
   (b) Find a basis for the column space.
   (c) Find a basis for the null space.

Solution:

(a) Notice that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix},$$

so the standard matrix of T is given by $M_T = \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

(b) If we reduce the matrix M_T, we obtain the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, the set of the first and third columns of M_T,

$$\left\{ \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}$$

forms a basis for its column space.

(c) To find a basis for the null space of M_T, notice that the system of linear equations $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions, and all of them are of the form

$$\mathbf{u} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c \in \mathbb{R}.$$

Therefore, the set

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

forms a basis for the null space of M_T.

6. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ ($n \geq 3$)?

   (a) All α such that $a_1 \geq 0$;

   (b) All α such that $a_1 + 3a_2 = a_3$;

   (c) All α such that $a_2 = a_1^2$;

   (d) All α such that a_2 is rational.

Solution: By definition (see Section 3.5 in Poole's book), a subspace of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:

1. The zero vector $\mathbf{0}$ is in S.
2. S is closed under linear combinations, i.e., if $\mathbf{u}$ and $\mathbf{v}$ are in S and c is a scalar, then $c\mathbf{u} + \mathbf{v}$ is in S.

We use these properties to determine which of the given subsets of vectors are subspaces of $\mathbb{R}^n$.

(a) This is not a subspace of $\mathbb{R}^n$. For $\alpha = (a_1, a_2, \ldots, a_n)$, the element $(-1)\alpha = (-a_1, -a_2, \ldots, -a_n)$ is not in the subset because $-a_1 \leq 0$.

(b) This is a subspace of $\mathbb{R}^n$. Notice that the zero vector $\mathbf{0}$ is in the subset since $0 + 3(0) = 0$. For $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ in the subset, we have $a_1 + 3a_2 = a_3$ and $b_1 + 3b_2 = b_3$. Let c be a scalar. Then

$$\mathbf{u} = c\alpha + \beta = (ca_1 + b_1, ca_2 + b_2, ca_3 + b_3, \ldots, ca_n + b_n) = (u_1, u_2, u_3, \ldots, u_n).$$

Since

$$u_1 + 3u_2 = (ca_1 + b_1) + 3(ca_2 + b_2) = c(a_1 + 3a_2) + 3(b_1 + 3b_2) = ca_3 + b_3 = u_3,$$

we conclude that $c\alpha + \beta$ is in the subset.

(c) This is not a subspace of $\mathbb{R}^n$. If $\alpha = (a_1, a_2, \ldots, a_n)$ is in the set, we have $a_2 = a_1^2$. For a scalar c, $c\alpha = (ca_1, ca_2, \ldots, ca_n)$. Notice that

$$(ca_1)^2 = c^2 a_1^2 \neq c a_1^2 = ca_2,$$

so $c\alpha$ is not in the subset.

(d) This is not a subspace of $\mathbb{R}^n$. If a_2 is rational and $c = \pi$, then ca_2 is irrational, so $c\alpha$ is not in the subset.