Name: Date: 2/9/2023

**Math 20580 L.A. and D.E. Tutorial**
**Worksheet 4**

1. Given the matrix $A = \begin{bmatrix} 1 & -1 & -2 & -3 \\ -1 & 1 & 1 & 2 \\ -1 & 1 & -1 & 12 \end{bmatrix}$, find a basis for Row(A), Col(A), Null(A).

   What are dimensions of Row(A), Col(A) and Null(A)?

   Hint: the first step is to row reduce A. Then, the non-zero rows will form a basis of Row(A), and pivots will indicate which columns of A form a basis of Col(A) (but we do not pick columns of a REF of A for a basis of Col(A)!). For Null(A), augment A by zero and solve the resulting system.

***Solution:*** Row reduce A

$$\begin{bmatrix} 1 & -1 & -2 & -3 \\ -1 & 1 & 1 & 2 \\ -1 & 1 & -1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

We see that a basis for Row(A) is $\{[1 \ -1 \ -2 \ -3], \ [0 \ 0 \ -1 \ 5], \ [0 \ 0 \ 0 \ 12]\}$. Thus, dimension of Row(A) is 3, which is number of nonzero rows in REF of A. A basis for Col(A) can be chosen as the first, the third and the forth columns of A: $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 12 \end{bmatrix} \right\}$.

Dimension of the Col(A) is 3, same as for Row(A). Thus, dimension of Null(A) is 4-3=1 (Rank Thm), or you can find it from a basis of Null(A). For the null space, augment A by zero and solve $\left[\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 12 & 0 \end{array}\right]$. The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, a basis for Null(A) can be chosen as $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

2. Let $\mathbf{b_1} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, $\mathbf{b_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; $\mathbf{c_1} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ be two bases for $\mathbb{R}^2$, find $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

***Solution:*** To solve two systems simultaneously, augment the coefficient matrix by $\mathbf{b_1}$ and $\mathbf{b_2}$:

$$\left[\begin{array}{cc|cc} 1 & 3 & -6 & 3 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{27}{7} & \frac{-12}{7} \\ 0 & 1 & \frac{-23}{7} & \frac{11}{7} \end{array}\right]$$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{27}{7} & \frac{-12}{7} \\ \frac{-23}{7} & \frac{11}{7} \end{bmatrix}.$$

3. Let $\mathcal{B}$ denote the basis of $\mathbb{R}^3$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v}$ denote the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Find the coordinates $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$.

***Solution:***

We have to solve the linear system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}\right].$$

Therefore $[\mathbf{v}]_{\mathcal{B}}$ is $\begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}$.

4. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for $\mathbb{R}^2$. Also suppose that the change-of-coordinate matrix **from** $\mathcal{B}$ **to** $\mathcal{C}$ is given as $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$. For $\mathbf{v} = \mathbf{b}_1 - 3\mathbf{b}_2$, what is $[\mathbf{v}]_\mathcal{C}$, the $\mathcal{C}$-coordinate for $\mathbf{v}$?

***Solution:***

$\mathbf{v} = \mathbf{b}_1 - 3\mathbf{b}_2$ means that $[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

So

$$[\mathbf{v}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

5. Find $\mathcal{C}$ if $\mathcal{B} = \left\{\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ from Question 4.

***Solution:***

We want to find the vectors $\mathbf{c}_1, \mathbf{c}_2$ in standard coordinates. Here are two solutions:

*Solution 1*: Perhaps you started by interpreting what the change-of-coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is doing. The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ tells us that

- $\mathbf{b}_1 = 3\mathbf{c}_1 + 5\mathbf{c}_2$, and

  (Reason: $[\mathbf{b}_1]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{b}_1]_\mathcal{B} = P_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.)

- $\mathbf{b}_2 = \mathbf{c}_1 + 2\mathbf{c}_2$.

  (Reason: $[\mathbf{b}_2]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{b}_2]_\mathcal{B} = P_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.)

Now this is not super helpful since we are given $\mathbf{b}_1, \mathbf{b}_2$ and we are trying to find $\mathbf{c}_1, \mathbf{c}_2$, but the inverse of $P_{\mathcal{C}\leftarrow\mathcal{B}}$, which is the change-of-coordinate matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$, will give us $\mathbf{c}_1, \mathbf{c}_2$ as linear combinations of $\mathbf{b}_1, \mathbf{b}_2$, i.e.

$$\mathbf{c}_1 = ?\mathbf{b}_1 + ?\mathbf{b}_2 \quad \text{and} \quad \mathbf{c}_2 = ?\mathbf{b}_1 + ?\mathbf{b}_2.$$

We compute the inverse of $P_{\mathcal{C}\leftarrow\mathcal{B}}$:

$$(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{3(2) - 1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

Therefore,

- $\mathbf{c}_1 = 2\mathbf{b}_1 - 5\mathbf{b}_2 = 2\begin{bmatrix}0\\3\end{bmatrix} - 5\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix}$.
- $\mathbf{c}_2 = -\mathbf{b}_1 + 3\mathbf{b}_2 = -1\begin{bmatrix}0\\3\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$.

*Solution 2:* We know that $[\mathbf{c}_1]_{\mathcal{C}} = \begin{bmatrix}1\\0\end{bmatrix}$, and that $\mathbf{c}_1 = [\mathbf{c}_1]_{std} = P_{\text{std}\leftarrow\mathcal{C}}\ [\mathbf{c}_1]_{\mathcal{C}}$, so we would be almost done if we could find the change-of-coordinate matrix $P_{\text{std}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to *std*.

Since $\mathcal{B} = \left\{\begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, we have the change-of-coordinate matrix from $\mathcal{B}$ to *std*:

$$P_{\text{std}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}.$$

We are also given the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$

So,

$$P_{\text{std}\leftarrow\mathcal{C}} = P_{\text{std}\leftarrow\mathcal{B}}\ P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\text{std}\leftarrow\mathcal{B}}\ \left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}.$$

We compute the inverse of $P_{\mathcal{C}\leftarrow\mathcal{B}}$:

$$\left(P_{\mathcal{B}\leftarrow\mathcal{C}}\right)^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{3(2)-1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

Now we can compute $P_{\text{std}\leftarrow\mathcal{C}}$:

$$P_{\text{std}\leftarrow\mathcal{C}} = P_{\text{std}\leftarrow\mathcal{B}}\ \left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}.$$

Finally, we can find $\mathbf{c}_1$:

$$\mathbf{c}_1 = P_{\text{std}\leftarrow\mathcal{C}}\ [\mathbf{c}_1]_{\mathcal{C}} = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix},$$

which is the first column of $P_{\text{std}\leftarrow\mathcal{C}}$.

Similarly, $\mathbf{c}_2$ is

$$\mathbf{c}_2 = P_{\text{std}\leftarrow\mathcal{C}}\ [\mathbf{c}_2]_{\mathcal{C}} = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix},$$

which is the second column of $P_{\text{std}\leftarrow\mathcal{C}}$.

6. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, find $\mathbf{x}$.

***Solution:***

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ means that the coordinates of $\mathbf{x}$ relative to the $\mathcal{B}$ basis is $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, so

$$\mathbf{x} = -1 \cdot \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 10 \end{bmatrix}.$$