Name: SOLUTIONS Date: 02/16/2023

M20580 L.A. and D.E. Tutorial
Worksheet 5

1. Determine whether the statements are true or false and justify your answer.

   (a) A set containing a single vector is linearly independent.

   (b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

   (c) Every linearly dependent set contains the zero vector.

   (d) The span of $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is the column space of the matrix whose column vectors are $\mathbf{v_1}, \ldots, \mathbf{v_n}$

   (e) The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

   (f) The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is not in the column space of A.

Solution: (a) F: $\{\mathbf{0}\}$ is linearly dependent. Any set containing a single nonzero vector is linearly independent.

(b) T : $-k\mathbf{v} + k\mathbf{v} = 0$.

(c) F : $\{\mathbf{v}, k\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ and $k \neq 0$.

(d) T: by definition.

(e) F: The column space of an $m \times n$ matrix A is the span of the columns. Equivalently, it is the set of all vectors $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

(f) T : If $A\mathbf{x} = \mathbf{b}$ is inconsistent, then there does not exist a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. Therefore, $\mathbf{b}$ is not a linear combination of the columns of A, hence not in the column space of A.

If $\mathbf{b}$ is not in the column space of A, then $\mathbf{b}$ is not a linear combination of the columns of A. So, there does not exist a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$, hence $A\mathbf{x} = \mathbf{b}$ is inconsistent.

2. Show that the vectors $\mathbf{v_1} = (-2, 0, 2), \mathbf{v_2} = (-1, -2, -1), \mathbf{v_3} = (0, 3, -2)$ form a basis of $\mathbb{R}^3$.

Solution: The matrix $\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & 3 \\ 2 & -2 & -2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So, $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent and span all of $\mathbb{R}^3$.

3. Test the following sets of polynomials for linear independence. For those that are linearly dependent, express one of the polynomials as a linear combination of the others.

(a) $\{x, 1+x\}$ in $\mathscr{P}_1$

(b) $\{x, 2x - x^2, 3x + 2x^2\}$ in $\mathscr{P}_2$

(c) $\{1 - 2x, 3x + x^2 - x^3, 3 + 2x + 3x^3\}$ in $\mathscr{P}_3$

Solution: (a) Recall that a set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly dependent if and only if we can find some scalars $a_1, \ldots, a_n$, with at least one being nonzero, such that $a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = 0$. Thus we consider the linear combination

$$a(x) + b(1+x) = 0$$

and solve for a and b. We can rewrite this as

$$(a+b)x + b = 0x + 0,$$

and we get the following system of equations (by equating coefficients of the same x-power term):

$$\begin{cases} a+b &= 0 \\ b &= 0 \end{cases}$$

In this case it is not necessary to write the augmented matrix and put it into RREF as it is clear that $b = 0$ and thus $a = 0$. From this we conclude that the set is linearly independent.

(b) Similar to (a), we consider the linear combination

$$a(x) + b(2x - x^2) + c(3x + 2x^2) = 0,$$

which can be rewritten as

$$(-b + 2c)x^2 + (a + 2b + 3c)x + 0 = 0x^2 + 0x + 0.$$

From this we get the following system of equations:

$$\begin{cases} -b + 2c &= 0 \\ a + 2b + 3c &= 0 \\ 0 &= 0 \end{cases}$$

with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this we see that c is free, $b = 2c$, and $a = -2b - 3c = -2(2c) - 3c = -7c$. Choosing $c = 1$, we have $b = 2$ and $a = -7$ as one possible solution. Thus the set is linearly dependent, with

$$-7(x) + 2(2x - x^2) + 1(3x + 2x^2) = 0.$$

So

$$3x + 2x^2 = 7(x) - 2(2x - x^2).$$

(c) Similar to (a) and (b), we look at the linear combination

$$a(1 - 2x) + b(3x + x^2 - x^3) + c(1 + x^2 + 2x^3) + d(3 + 2x + 3x^3) = 0$$

and get the following augmented matrix from the corresponding system of equations:

$$\left[\begin{array}{cccc|c} 0 & -1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 0 & 2 & 0 \\ 1 & 0 & 1 & 3 & 0 \end{array}\right].$$

Using row operations to reduce the matrix into RREF, we can conclude that $a = b = c = d = 0$. Thus the set is linearly independent.

4. $V = \mathscr{P}_3$ is a vector space, and $S = \{x^3 - 2x^2 + 3x - 1, 2x^3 + x^2 + 3x - 2\}$ is a subset of V. Determine if the given vector $\mathbf{v} = -2x^3 - 11x^2 + 3x + 2$ of V is in span(S).

   If it is, find an explicit representation of $\mathbf{v}$ as a linear combination of the vectors in S.

Solution: The equation we are solving is

$$\begin{aligned} -2x^3 - 11x^2 + 3x + 2 &= a\left(x^3 - 2x^2 + 3x - 1\right) + b\left(2x^3 + x^2 + 3x - 2\right) \\ &= (a+2b)x^3 + (-2a+b)x^2 + (3a+3b)x + (-a-2b) \end{aligned}$$

This is equivalent to the following system linear equations (by equating coefficients of same x-power term):

$$\begin{cases} a + 2b &= -2 \\ -2a + b &= -11 \\ 3a + 3b &= 3 \\ -a - 2b &= 2 \end{cases}$$

We have

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 5 & -15 \\ 0 & -3 & 9 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

The system has solution $a = 4$ and $b = -3$, so that

$$-2x^3 - 11x^2 + 3x + 2 = 4\left(x^3 - 2x^2 + 3x - 1\right) - 3\left(2x^3 + x^2 + 3x - 2\right).$$

Thus $\mathbf{v}$ is in span(S).

5. All of the following sets with their operations are not vector spaces. State at least one axiom of vector spaces that does not hold in each case and justify your answer with concrete examples:

(a) The set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ with usual vector addition and scalar multiplication.

(b) $\mathbb{R}^2$, with the usual scalar multiplication but addition is defined by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix}.$$

(c) The set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad = 0$ with usual matrix operations.

Solution: (a) We have $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector of the proposed form, but $k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$ is not in the considered set if $k < 0$, so this set is not closed under scalar multiplication (axiom 6 of vector spaces).

(b) Consider vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ with the scalar 2 we have

$$2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = 2\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

while

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \end{bmatrix}$$

so distributivity does not hold (axiom 7 of vector spaces).

(c) The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in the considered set, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not, so this set is not closed under vector addition (axiom 1 of vector spaces).