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M20580 L.A. and D.E. Tutorial
Worksheet 6

1. Consider the basis $\mathcal{B} = \left\{ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$ of $\mathcal{M}_{2\times 2}$. Find the coordinate vector of $A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ with respect to $\mathcal{B}$, i.e. the vector $[A]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $A = aM_1 + bM_2 + cM_3 + dM_4$.

Solution: (First method) Consider the equation

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = A = aM_1 + bM_2 + cM_3 + dM_4 = \begin{bmatrix} a+b+c+d & -b-c-d \\ -c-d & d \end{bmatrix}.$$

Then $d = 8, c = -13, b = 2$, and $a = 5$. Thus

$$[A]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ -13 \\ 8 \end{bmatrix}.$$

(Second method) Recall that

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = A = aM_1 + bM_2 + cM_3 + dM_4 = \begin{bmatrix} a+b+c+d & -b-c-d \\ -c-d & d \end{bmatrix}.$$

This equivalents to the following system of equations

$$\begin{cases} a & +b & +c & +d & = & 2 \\ & -b & -c & -d & = & 3 \\ & & -c & -d & = & 5 \\ & & & d & = & 8 \end{cases}.$$

That is we have the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

Thus $d = 8, c = -13, b = 2$, and $a = 5$.

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2. In each of the following, V is a vector space and W is a subset of V. Determine if W is a subspace of V. Justify your answer.

   (a) $V = \mathbb{R}^3$ and $W = \left\{ \begin{bmatrix} u \\ -\pi u \\ 0 \end{bmatrix} | u \in \mathbb{R} \right\}$,

   (b) $V = \mathcal{P}_3$ and $W = \{a + bx + cx^2 + dx^3 | ab = cd\}$,

   (c) $V = \mathcal{M}_{2\times 2}(\mathbb{R})$ and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R}, ad \geq 0 \right\}$.

Solution:

(a) Since $W = \text{span}\{ \begin{bmatrix} 1 \\ -\pi \\ 0 \end{bmatrix} \}$, it is a subspace.

(b) Consider $p(x) = 1$ and $q(x) = x$. We have $p(x), q(x) \in W$ but $p(x) + q(x) = 1 + x \notin W$. Hence, W is not a subspace.

(c) Consider $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. We have $p, q \in W$ but $p + q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \notin W$. Thus W is not a subspace.

Recall A *vector space* is a nonempty set V with objects *vectors* and two operations, addition and multiplication by scalars, such that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalar c and d

1. $\mathbf{u} + \mathbf{v} \in V$ (cloesure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of addition)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity of addition)
4. there exists an element $\mathbf{0}$ in V, the *zero* vector such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. for any $\mathbf{u} \in V$ there exists an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u} \in V$ (closure under scalar multiplication)
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (associativity of scalar multiplication)
8. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity)
9. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity)
10. $1\mathbf{u} = \mathbf{u}$.

3. Let $M = \{f(x) = ae^{2x} + be^{-2x} | a, b \in \mathbb{R}\}$. Define $T : M \to \mathbb{R}^2$ by

$$T(f(x)) = \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}.$$

(a) Show that M is a vector space with the usual addition and scalar multiplication.

(b) Show that T is a linear transformation.

(c) Find the kernel and range of T. Show that M is isomorphic to $\mathbb{R}^2$.

Solution:

(a) For $i = 1, 2, 3$, let $f_i = a_i e^{2x} + b_i e^{-2x} \in M$ and c, d be scalar. Then

1. $f_1 + f_2 = (a_1 + a_2)e^{2x} + (b_1 + b_2)e^{-2x} \in M$
2. $f_1 + f_2 = (a_1 + a_2)e^{2x} + (b_1 + b_2)e^{-2x} = (a_2 + a_1)e^{2x} + (b_2 + b_2)e^{-2x} = f_2 + f_1$
3. $$\begin{aligned} f_1 + (f_2 + f_3) &= (a_1 + (a_2 + a_3))\, e^{2x} + (b_1 + (b_2 + b_3))\, e^{-2x} \\ &= ((a_1 + a_2) + a_3)\, e^{2x} + ((b_1 + b_2) + b_3)\, e^{-2x} \\ &= (f_1 + f_2) + f_3 \end{aligned}$$
4. consider $\mathbf{0} = 0$ Then $f_1 + \mathbf{0} = f_1 + 0 = f_1$
5. consider $f_1 + (-f_1) = f_1 - f_1 = 0 = \mathbf{0}$
6. $cf_1 = ca_1 e^{2x} + cb_1 e^{-2x} \in M$
7. $c(df_1) = c(da_1 e^{2x} + db_1 e^{-2x}) = cda_1 e^{2x} + cdb_1 e^{-2x} = (cd)f_1$
8. $c(f_1 + f_2) = c(a_1 + b_1)e^{2x} + c(b_1 + b_2)e^{-2x} = cf_1 + cf_2$
9. $(c + d)f_1 = (c + d)a_1 e^{2x} + (c + d)b_1 e^{-2x} = cf_1 + df_1$
10. $1f_1 = f_1$.

(b) Let $c \in \mathbb{R}$ and $f(x), g(x) \in M$.

$$T(cf(x) + g(x)) = \begin{bmatrix} cf(0) + g(0) \\ cf'(0) + g'(0) \end{bmatrix} = cT(f(x)) + T(g(x)).$$

Then T is linear.

(c) Let $f(x) = ae^{2x} + be^{-2x}$. Then

$$T(f(x)) = \begin{bmatrix} a + b \\ 2a - 2b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix},$$

then $\text{Ker}(T) = \{0\}$ and $\text{Range}(T) = \mathbb{R}^2$. Then T is a bijective linear transformation from M to $\mathbb{R}^2$. Hence M is isomorphic to $\mathbb{R}^2$.

4. Determine whether the set $\mathcal{B}$ is a basis for the vector space V.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ and $V = \mathcal{M}_{2\times 2}$.

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ and $V = \{A \in \mathcal{M}_{2\times 2} | A^T = A\}$,

(c) $\mathcal{B} = \{1 + x^2, 1 + 2x + 3x^2\}$ and $V = \{a + bx + cx^2 | a + b = c \text{ and } a, b, c \in \mathbb{R}\}$

Solution:

(a) Note that $\dim(V) = 4$. We will show that $\mathcal{B}$ is linearly dependent. Consider

$$\mathbf{0} = a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+d & a+b \\ c+d & b+c \end{bmatrix}.$$

We have the following system of equations

$$\begin{cases} a+d &= 0 \\ a+b &= 0 \\ c+d &= 0 \\ b+c &= 0 \end{cases}.$$

That is we have the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\mathcal{B}$ is not a basis.

(b) Note that $A^T = A$ implies that $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ where $a, b, c \in \mathbb{R}$, i.e. $\dim(V) = 3$. We will show that $\mathcal{B}$ is linearly independent. Consider

$$\mathbf{0} = a \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2a-b & c \\ c & 2b-a \end{bmatrix}$$

Then $c = 0$ and the following augmented matrix for a and b

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1.5 & 0 \end{array}\right]$$

Thus, $\mathcal{B}$ is a basis for V.

(c) Note that for any $f(x) = a + bx + cx^2 \in V$, we have $f(x) = a + bx + (a+b)x^2 = a + ax^2 + bx + bx^2$ where $a, b \in \mathbb{R}$, i.e. $\dim(V) = 2$. Consider

$$0 = a(1 + x^2) + b(1 + 2x + 3x^2) = a + b + 2bx + (a + 3b)x^2.$$

Then $b = 0$ and $a = 0$. Thus $\mathcal{B}$ is a basis for V.

5. Let $\mathcal{E} = \{e^{2x}, e^{-2x}\}$ and $M = \{f(x) = ae^{2x} + be^{-2x} | a, b \in \mathbb{R}\} = \operatorname{span} \mathcal{E}$. Consider a basis $\mathcal{B} = \{\sinh(2x), \cosh(2x)\}$ of M, where

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}, \quad \cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}.$$

   (a) Find the change of basis matrix $P_{\mathcal{E} \leftarrow \mathcal{B}}$.

   (b) Find the matrices $[D]_{\mathcal{E}}$ and $[D]_{\mathcal{B}}$ of the linear transformation $D : M \to M$ given by $D(f(x)) = f''(x)$.

Solution:

(a) In terms of $\mathcal{E}$, the basis $\mathcal{B}$ is

$$\mathcal{B} = \{ \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \}.$$

Then the change of basis matrix

$$P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(b) Recall that

$$[D]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}} [D]_{\mathcal{E}} P_{\mathcal{E} \leftarrow \mathcal{B}}.$$

We have

$$(P_{\mathcal{E} \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Now, the operator D acts upon e^{2x} and e^{-2x} as

$$D(e^{2x}) = 4e^{2x}, \quad D(e^{-2x}) = 4e^{-2x}.$$

Then

$$[D]_{\mathcal{E}} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus,

$$[D]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}} [D]_{\mathcal{E}} P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$