Name: SOLUTIONS Date: 03/02/2023

M20580 L.A. and D.E. Tutorial
Worksheet 7

1. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 + 2x_2 \end{bmatrix}.$$

(a) What is the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

(b) If $B = \{\vec{\alpha_1}, \vec{\alpha_2}, \vec{\alpha_3}\}$ and $\mathcal{C} = \left\{\vec{\beta_1}, \vec{\beta_2}\right\}$, where

$$\vec{\alpha_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{\alpha_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{\alpha_3} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{\beta_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{\beta_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

what is the matrix of $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution: (a) Since $T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the matrix of T is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

(b) First, $T\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Next, $T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Lastly, $T\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. So, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is $\begin{bmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

2. Define a linear transformation $T : \mathcal{P}_2 \to \mathbb{R}^2$ via

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

Let $\mathcal{B} = \{1, 1+x, x+x^2\}$ be another basis for $\mathcal{P}_2$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \end{bmatrix} \right\}$ be another basis for $\mathbb{R}^2$. Compute $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$

Hint: you may want to start with $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[T(1)]_{\mathcal{C}} \vdots [T(1+x)]_{\mathcal{C}} \vdots \left[T\left(x+x^2\right)\right]_{\mathcal{C}} \right]$; for this approach, you will also need to compute $P_{\mathcal{C}\leftarrow\mathcal{E}}$. Another, more computationally demanding way would be to use the formula $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}}[T]_{\mathcal{E}\leftarrow\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}}$.

Solution: The matrix $P_{\mathcal{C}\leftarrow\mathcal{E}}$ is given by

$$P_{\mathcal{C}\leftarrow\mathcal{E}} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$$

so

$$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$
$$[T(1+x)]_{\mathcal{C}} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$
$$\left[T\left(x+x^2\right)\right]_{\mathcal{C}} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

Thus

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 8 & 13 & 10 \\ -3 & -5 & -4 \end{bmatrix}$$

3. Let $T : M_{2\times 2} \to M_{2\times 2}$ be a linear transformation defined by

$$T(\mathbf{A}) := \mathbf{A}\mathbf{W_0} - \mathbf{W_0}\mathbf{A}, \quad \text{where} \quad \mathbf{W_0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

a) Find the matrix $[T]_{\mathcal{E}}$ of T in the standard basis

$$\mathcal{E} = \left\{ \mathbf{E_1} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{E_2} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{E_3} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{E_4} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

b) Consider a basis $\mathcal{B}$ in $M_{2\times 2}$ given by

$$\mathcal{B} := \left\{ \mathbf{B_1} := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \mathbf{B_2} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{B_3} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{B_4} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right\}.$$

Compute $[T]_{\mathcal{B}}$

c) Use $[T]_{\mathcal{B}}$ to compute the null space of T.

Hint: for part b), it would be easier if instead of using the formula $[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[T]_{\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}}$ you use $[T]_{\mathcal{B}} = \left[[T(\mathbf{B_1})]_{\mathcal{B}} \vdots [T(\mathbf{B_2})]_{\mathcal{B}} \vdots [T(\mathbf{B_3})]_{\mathcal{B}} \vdots [T(\mathbf{B_4})]_{\mathcal{B}} \right]$. I.e., try acting by T upon the matrices from $\mathcal{B}$, see how the outputs can be expressed as linear combinations of $\mathcal{B}$, and collect the coefficients into the columns of $[T]_{\mathcal{B}}$. Once you learn a bit more about linear algebra, you'll be able to see that $\mathcal{B}$ is a so-called eigen-basis of T.

Solution: The action of T upon $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$T(\mathbf{A}) = \begin{bmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{bmatrix} - \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = \begin{bmatrix} a_{12} - a_{21} & a_{11} - a_{22} \\ a_{22} - a_{11} & a_{21} - a_{12} \end{bmatrix}.$$

Therefore, the matrix of T in $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Now, let's find $[T]_{\mathcal{B}}$ via $[T]_{\mathcal{B}} = \left[[T(\mathbf{B_1})]_{\mathcal{B}} \vdots [T(\mathbf{B_2})]_{\mathcal{B}} : [T(\mathbf{B_3})]_{\mathcal{B}} \vdots [T(\mathbf{B_4})]_{\mathcal{B}} \right]$:

$$T(\mathbf{B_1}) = -2\mathbf{B_1}, \quad T(\mathbf{B_2}) = 0 \cdot \mathbf{B_2}, \quad T(\mathbf{B_3}) = 0 \cdot \mathbf{B_3}, \quad T(\mathbf{B_4}) = 2\mathbf{B_4}.$$

Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Now, the null space of T can be easily read off from $[T]_{\mathcal{B}}$:

$$\text{null}(T) = \text{span}\,\{\mathbf{B_2}, \mathbf{B_3}\}$$

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4. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{bmatrix}$.

   (a) Compute the first column of the cofactor matrix associated to $\mathbf{A}$.

   (b) Using part (a) compute the determinant of $\mathbf{A}$.

Solution: (a) We compute C_{11}, C_{21}, and C_{31}.

$$C_{11} = (-1)^{1+1} \det\left(\begin{bmatrix} 3 & -2 \\ 3 & 0 \end{bmatrix}\right) = (-1)^{1+1}((3)(0) - (-2)(3)) = 6.$$
$$C_{21} = (-1)^{2+1} \det\left(\begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}\right) = (-1)^{2+1}((1)(0) - (-1)(3)) = -3.$$
$$C_{31} = (-1)^{3+1} \det\left(\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}\right) = (-1)^{3+1}((1)(-2) - (-1)(3)) = 1.$$

(b) Using expansion along the first column, the determinant is $0C_{11} + 2C_{21} - 1C_{31} = 0(6) + 2(-3) - 1(1) = -7$.

5. Consider the matrix $\mathbf{A} = \begin{bmatrix} t & -t & 0 \\ 6 & 5t & t^2 \\ 0 & t & -t \end{bmatrix}$, where t is some real number.

(a) Find the determinant of $\mathbf{A}$.

(b) Find all values of t for which $\mathbf{A}$ is invertible. (Recall that a square matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.)

Solution: (a) Computing a cofactor expansion along the first row yields:

$$\begin{aligned} \det(\mathbf{A}) &= t \begin{vmatrix} 5t & t^2 \\ t & -t \end{vmatrix} - (-t) \begin{vmatrix} 6 & t^2 \\ 0 & -t \end{vmatrix} \\ &= t\left(-5t^2 - t^3\right) + t\,(-6t) \\ &= -t^4 - 5t^3 - 6t^2. \end{aligned}$$

(b) We need to find the roots of $-t^4 - 5t^3 - 6t^2$. Factoring, we have

$$-t^4 - 5t^3 - 6t^2 = -t^2\left(t^2 + 5t + 6\right) = -t^2(t+2)(t+3).$$

The roots are $0, -2, -3$. So, $\mathbf{A}$ is invertible when $t \neq 0, -2, -3$.

6. Use Cramer's rule to find a solution to the following system of equations:

$$\begin{aligned} x + y - z &= 1 \\ x + y + z &= 2 \\ x - y &= 3 \end{aligned}$$

Solution: We can write the above equation in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Replacing the first column with the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we obtain: $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix}$. This matrix has determinant 9 . The original matrix has determinant 4. So $x = \frac{9}{4}$

Replacing the second column with the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we obtain: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$. This matrix has determinant -3. The original matrix has determinant 4. So $y = -\frac{3}{4}$

Replacing the third column with the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we obtain: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$. This matrix has determinant 2. The original matrix has determinant 4. So $z = \frac{2}{4} = \frac{1}{2}$

Finally, we can check that $\begin{bmatrix} \frac{9}{4} \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$ is a solution: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{9}{4} \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.