Name: SOLUTIONS Date: 03/09/2023

M20580 L.A. and D.E. Tutorial
Worksheet 8

1. For both parts (a) and (b) of this problem, show that $\mathbf{v}$ is an eigenvector of A and find the corresponding eigenvalue

   (a) $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Solution: (a) Recall that $\mathbf{v}$ is an eigenvector of A if there exists a $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. So we just need to apply A to $\mathbf{v}$ to figure out what the λ is.

$$A\mathbf{v} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3 \cdot \mathbf{v}$$

thus $\mathbf{v}$ is an eigenvector with $\lambda = -3$.

(b) Recall that $\mathbf{v}$ is an eigenvector of A if there exists a $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. So we just need to apply A to $\mathbf{v}$ to figure out what the λ is.

$$A\mathbf{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 3 \cdot \mathbf{v}$$

thus $\mathbf{v}$ is an eigenvector with $\lambda = 3$.

Name: SOLUTIONS Date: 03/09/2023

2. For both parts (a) and (b) of this problem, show that λ is an eigenvalue of A and find at least one corresponding eigenvector.

   (a) $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, $\lambda = 3$

   (b) $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, $\lambda = -1$

Solution: (a) We start by finding the matrix $A - \lambda I$:

$$A - \lambda I = A - 3I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

The REF of $A - 3I$ is:

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

This tells us that a vector of the form $\mathbf{v} = s \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where $s \in \mathbb{R}$, is in the null space of $A - 3I$ and is thus an eigenvector of the matrix A with eigenvalue $\lambda = 3$.

(b) We start by finding the matrix $A - \lambda I$:

$$A - \lambda I = A + I = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

The REF of $A + I$ is:

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This tells us that a vector of the form $\mathbf{v} = s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, where $s \in \mathbb{R}$, is in the null space of $A + I$ and is thus an eigenvector of the matrix A with eigenvalue $\lambda = -1$.

3. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

For each matrix given, go through the following steps:

i. Find the characteristic polynomial
ii. Determine the eigenvalues with corresponding algebraic multiplicities
iii. Find a basis for the eigenspace of each eigenvalue
iv. Determine the geometric multiplicity of each eigenvalue

Solution: For matrix A, we start by finding the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) + 6 = (\lambda - 4)(\lambda - 3)$$

This immediately gives us that the eigenvalues are 3 and 4 with both having algebraic multiplicity 1. We now find the eigenvectors by finding the null space of $A - \lambda I$ with $\lambda = 3, 4$.

$$A - 3I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$$
$$A - 4I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}$$

One can immediately see that the null space of $A - 3I$ (i.e. the eigenspace of $\lambda = 3$) is spanned by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ which tells us that the geometric multiplicity of $\lambda = 3$ is 1. Furthermore, the eigenspace of $\lambda = 4$ is clearly spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so it too has a geometric multiplicity of 1.

Name: SOLUTIONS Date: 03/09/2023

For matrix B, we start by finding the characteristic polynomial:

$$\det(B - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)(2-\lambda) + (3-\lambda) = (3-\lambda)^3$$

This immediately gives us that the only eigenvalue is 3 with an algebraic multiplicity of 3. We now find the eigenvectors by finding the null space of $A - 3I$.

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

One can immediately see that the null space of $A - 3I$ (i.e. the eigenspace of $\lambda = 3$) is spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ which tells us that the geometric multiplicity of $\lambda = 3$ is 2.

4. Explain why the following matrices are not similar (**Hint:** recall that if two matrices are similar their characteristic polynomials must be equal).

   (a) $A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

Solution: (a) We show that the characteristic polynomials are not equal.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16$$

$$\neq \lambda^2 - 10\lambda + 16 = (3-\lambda)(7-\lambda) - 5 = \begin{vmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{vmatrix} = \det(B - \lambda I)$$

Therefore, the two matrices cannot be similar.

(b) Based on what you have learned in class, you should immediately see that these are triangular matrices; therefore, their characteristic polynomials are just the products along the diagonal.

$$\det(A - \lambda I) = (2-\lambda)^2(4-\lambda) \neq (1-\lambda)(4-\lambda)^2 = \det(B - \lambda I)$$

Since their characteristic polynomials are not equal, we get that the two matrices cannot be similar

5. Consider the matrix:

$$A = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$$

and follow these steps:

i. Find the characteristic polynomial and determine the eigenvalues
ii. Find a basis for each eigenspace corresponding to each eigenvalue
iii. Find the matrices P and D such that D is diagonal and $A = PDP^{-1}$
iv. Find P^{-1} and explicity compute PDP^{-1} to confirm it is equal to A.

Solution: We start by finding the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 6 \\ 1 & -\lambda \end{vmatrix} &= \lambda(1+\lambda) - 6 \\ &= (\lambda - 2)(\lambda + 3) \end{aligned}$$

So we have 2 distinct eigenvalues: $2, -3$. Since A is a 2×2 matrix with 2 distinct eigenvalues, we know by theorem 4.25 in the book that A is diagonalizable, and by theorem 4.27 that each eigenspace will be spanned by a single vector, i.e. the null space of $A - \lambda I$ will be spanned by a single vector. Recalling that we denote eigenspaces by E_λ, we now find each of these vectors in turn:

$$\begin{aligned} A - 2I &= \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} \Rightarrow E_2 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \\ A + 3I &= \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \Rightarrow E_{-3} = \text{span}\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\} \end{aligned}$$

We now need to construct the P and D matrices. Recall that P is just the change of basis matrix from the basis of eigenvectors to the standard basis. We can then think of D as the transformation matrix in the basis of eigenvectors. In other words, denoting the basis of eigenvectors $\mathcal{C} = \left\{ \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ (remember: the order of the basis vectors matters here), we have:

$$\begin{aligned} P &= \begin{bmatrix} [\mathbf{v}]_{\mathcal{E}} & [\mathbf{w}]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \\ D &= \begin{bmatrix} [A\mathbf{v}]_{\mathcal{C}} & [A\mathbf{w}]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \end{aligned}$$

Recalling that the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, we get:

$$P^{-1} = -\frac{1}{5}\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

and you should verify that we do get:

$$A = -\frac{1}{5}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

6. Consider the transformation $\mathcal{D} : \mathcal{P}_2 \to \mathcal{P}_2$ which takes a polynomial $p(x)$ to $p'(x)$.

   i. Find the matrix of this transformation in the basis $\{1, x, x^2\}$.

   ii. Find the characteristic polynomial and determine the eigenvalues along with their algebraic multiplicities.

   iii. Find a basis for each eigenspace and determine the geometric multiplicities.

   iv. Determine if the matrix of the transformation is diagonalizable.

Solution: Calling the matrix of the transformation A, we have that:

$$A = \begin{bmatrix} \mathcal{D}(1) & \mathcal{D}(x) & \mathcal{D}(x^2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Given that this is a triangular matrix, it is easy to see that:

$$\det(A - \lambda I) = -\lambda^3$$

From which we get that the only eigenvalue is 0 and its algebraic multiplicity is 3.

Since $A - 0 \cdot I = A$ and A is already in REF, we immediately see that:

$$E_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

From which it follows that the geometric multiplicity of 0 is 1.

By theorem 4.27 in the book, we know that if A is diagonalizable it must be that for each eigenvalue the algebraic multiplicity must be equal to the geometric multiplicity; however, we have only one eigenvalue with geometric multiplicity strictly less than algebraic multiplicity. It follows that A is not diagonalizable.