Name: SOLUTIONS Date: 03/23/23

M20580 L.A. and D.E. Tutorial
Worksheet 9

1. Find a diagonalization of the following matrix

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}.$$

Solution: We have the characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^2 - 5\lambda - 14 = (\lambda + 2)(\lambda - 7).$$

Thus the eigenvalues are $-2, 7$. Next, we are going to find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that associate with $\lambda_1 = -2$ and $\lambda_2 = 7$, respectively. Since $(A + 2I)$ has REF

$$\begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix},$$

one can choose $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Similarly, since $(A - 7I)$ has REF

$$\begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix},$$

we can choose $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} 5 & 1 \\ -4 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}.$$

2. Find all eigenvalues and a eigenvector for each eigenvalue of the following matrices

(a) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Solution: The characteristic equation is

$$\det(A - \lambda I) = \lambda^2 - 2\lambda + 5.$$

Then the eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$

The REF of $A - (1 + 2i)I$ is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus the eigenvalue $\mathbf{v}_1$ associated with $\lambda_1 = 1 + 2i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and the eigenvalue $\mathbf{v}_2$ associated with $\lambda_2 = 1 - 2i = \overline{\lambda_1}$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix} = \overline{\mathbf{v}}_1$.

(b) $B = \begin{bmatrix} 1 & 5 & -4 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution: The characteristic equation is

$$\det(A - \lambda I) = (2 - \lambda)(\lambda^2 - 2\lambda + 5).$$

Then the real eigenvalue is $\lambda = 2$ and the complex eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$

The REF of $A - 2I$ is

$$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus the eigenvalue $\mathbf{v}_1$ associated with $\lambda_1 = 2$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For the complex eigenvalues, we have REF of $A - (1 - 2i)I$ is

$$\begin{bmatrix} 2 & -5i & 4i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the eigenvalue $\mathbf{v}_2$ associated with $\lambda_2 = 1-2i$ is $\begin{bmatrix} -2i \\ 0 \\ 1 \end{bmatrix}$ and the eigenvalue $\mathbf{v}_3$ associated with $\lambda_3 = 1+2i = \overline{\lambda_2}$ is $\begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix} = \overline{\mathbf{v}}_2$.

3. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

Solution: $\mathbf{u} \cdot \mathbf{v} = 1 - 2 + 0 = -1 = \mathbf{v} \cdot \mathbf{u}$.

(b) Find a unit vector in the direction of $\mathbf{u}$

Solution: $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{2}$. Hence, a unit vector in the direction of $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{u}$.

(c) Find a unit vector in the direction of $\mathbf{v}$

Solution: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{14}$. Hence, a unit vector in the direction of $\mathbf{v} = \frac{1}{\sqrt{14}}\mathbf{v}$.

(d) Find $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{-1}{2}\mathbf{u}$$
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{-1}{14}\mathbf{v}.$$

4. Determine if the given vectors form an orthogonal set

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Solution: Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the given vectors respectively. Then we have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= -3 + 2 + 1 = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= -3/2 - 2 + 7/2 = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1/2 - 4 + 7/2 = 0. \end{aligned}$$

Then they form an orthogonal set.

5. Determine if the given vectors form an orthogonal basis for $\mathbb{R}^2$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 9 \end{bmatrix}.$$

Solution:

(1) Proving directly: Clearly, they form an orthogonal set. We only need to check if they are linearly independent. That is

$$\begin{aligned} \begin{bmatrix} 3 & -6 \\ 2 & 9 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 \\ 2 & 9 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 \\ 0 & 13 \end{bmatrix}. \end{aligned}$$

(2) Using Theorem 5.1 in Poole: *If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is an orthogonal set of nonzero vector in $\mathbb{R}^n$, then these vectors are linearly independent.* Since our vectors are nonzero and form an orthogonal set, they are also linearly independent. Thus they form an orthogonal basis for $\mathbb{R}^2$.

6. Find the orthogonal complement $W^\perp$ of W in $\mathbb{R}^3$ where

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + y = z \,, y + z = 0 \right\}.$$

Hint: $(\text{null}A)^\perp = \text{col}A^T$ and $(\text{col}A)^\perp = \text{null}A^T$.

Solution: Note that $W = \text{null}\{\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\}$. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then $W^\perp = \text{col}A^T$. We have that

$$A^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus $W^\perp = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.