Solutions to Math 54 Problem Set #9

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4.1.2 (a) If y is a solution, then $my'' + by' + ky = 0$. Multiplying by c we get

$$0 = c(my'' + by' + ky) = m(cy)'' + b(cy)' + k(cy) = 0,$$

so $c \cdot y$ is also a solution.

(b) If y_1, y_2 are solutions, then $my_1'' + by_1' + ky_1 = 0$ and $my_2'' + by_2' + ky_2 = 0$. Adding these relations together we get

$$0 = (my_1'' + by_1' + ky_1) + (my_2'' + by_2' + ky_2) = m(y_1 + y_2)'' + b(y_1 + y_2)' + k(y_1 + y_2) = 0,$$

so $y_1 + y_2$ is also a solution.

(a) and (b) together say that the set of solutions of the differential equation $my''+by'+ky = 0$ form a vector space.

4.1.3 We have

$$y' = 6\cos(3t) - 3\sin(3t),$$

and

$$y'' = -18\sin(3t) - 9\cos(3t),$$

so

$$2y'' + 18y = (-36\sin(3t) - 18\cos(3t)) + (36\sin(3t) + 18\cos(3t)) = 0.$$

We now check the initial conditions:

$$y(0) = 2\sin(0) + \cos(0) = 1,$$

$$y'(0) = 6\cos(0) - 3\sin(0) = 6.$$

The maximum of $|y(t)|$ is attained either at the maximal positive value of $y(t)$, or at the minimal negative value of $y(t)$. In any case, it is attained at an extremal value of $y(t)$, i.e. a point where $y'(t) = 0$. This condition is equivalent to $\sin(3t) = 2\cos(3t)$, i.e. $\tan(3t) = 2$. If this is the case, then

$$y(t) = \cos(3t)(2\tan(3t) + 1) = \frac{\pm 1}{\sqrt{1 + \tan^2(3t)}}(2\tan(3t) + 1) = \frac{\pm 1}{\sqrt{5}} \cdot 5 = \pm\sqrt{5},$$

so the maximal value of $|y(t)|$ is $\sqrt{5}$.

Alternatively, one can use the Cauchy-Schwarz inequality to get that

$$(y(t))^2 = (2\sin(3t) + \cos(3t))^2 \le (2^2 + 1^2) \cdot (\sin^2(3t) + \cos^2(3t)) = 5 \cdot 1 = 5,$$

with equality when $\sin(3t)/2 = \cos(3t)/1$, i.e. $\tan(3t) = 2$.

4.1.5 We have

$$y'(t) = -2e^{-2t}\sin(\sqrt{2}t) + \sqrt{2}e^{-2t}\cos(\sqrt{2}t) = e^{-2t}(-2\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t)),$$

and

$$\begin{aligned} y''(t) &= -2e^{-2t}(-2\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t)) + e^{-2t}(-2\sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t)) \\ &= e^{-2t}(2\sin(\sqrt{2}t) - 4\sqrt{2}\cos(\sqrt{2}t)). \end{aligned}$$

It follows that

$$\begin{aligned} my'' + by' + ky &= y'' + 4y' + 6y \\ &= e^{-2t}((2 + 4\cdot(-2) + 6)\sin(\sqrt{2}t) + (-4\sqrt{2} + 4\sqrt{2})\cos(\sqrt{2}t)) = 0. \end{aligned}$$

Since $|\sin(\sqrt{2}t)| \leq 1$ for all t, it follows that $0 \leq |y(t)| \leq e^{-2t}$, so by the Squeeze Theorem

$$0 \leq \lim_{t\to\infty} |y(t)| \leq \lim_{t\to\infty} e^{-2t} = 0,$$

i.e. $\lim_{t\to\infty} |y(t)| = 0$. It follows that $\lim_{t\to\infty} y(t) = 0$.

4.1.8 We look for solutions of the equation $y'' + 2y' + 4y = 5\sin(3t)$ of the form $y = A\cos(3t) + B\sin(3t)$. We have

$$\begin{aligned} y' &= -3A\sin(3t) + 3B\cos(3t), \\ y'' &= -9A\cos(3t) - 9B\sin(3t). \end{aligned}$$

The relation $y'' + 2y' + 4y = 5\sin(3t)$ becomes

$$(-9A + 6B + 4A)\cos(3t) + (-9B - 6A + 4B)\sin(3t) = 5\sin(3t),$$

i.e.

$$\begin{cases} -5A + 6B = 0 \\ -6A - 5B = 5 \end{cases}.$$

Solving this system we get $A = 30/61$, $B = 25/61$, so

$$y(t) = \frac{30}{61}\cos(3t) + \frac{25}{61}\sin(3t).$$

4.1.9 We look for solutions of the equation $y'' + 2y' + 4y = 3\cos(2t) + 4\sin(2t)$ of the form $y = A\cos(2t) + B\sin(2t)$. We have

$$\begin{aligned} y' &= -2A\sin(2t) + 2B\cos(2t), \\ y'' &= -4A\cos(2t) - 4B\sin(2t). \end{aligned}$$

The relation $y'' + 2y' + 4y = 3\cos(2t) + 3\sin(2t)$ becomes

$$(-4A + 4B + 4A)\cos(2t) + (-4B - 4A + 4B)\sin(2t) = 3\cos(3t) + 4\sin(2t),$$

i.e.

$$\begin{cases} 4B = 3 \\ -4A = 4 \end{cases}.$$

This gives $A = -1$, $B = 3/4$, so

$$y(t) = -\cos(2t) + \frac{3}{4}\sin(2t).$$

4.2.2 The auxiliary equation $r^2 - r - 2 = 0$ has distinct solutions $r_1 = 2$ and $r_2 = -1$. It follows that the general solution of the equation $y'' - y' - y = 0$ is given by

$$y(t) = c_1 e^{2t} + c_2 e^{-t}.$$

4.2.15 The auxiliary equation $r^2 + 2r + 1 = 0$ has a double solution $r_1 = r_2 = -1$. It follows that the general solution of the equation $y'' + 2y' + y = 0$ is given by

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

The condition $y(0) = 1$ translates into $c_1 = 1$, while the condition $y'(0) = -3$ yields

$$-c_1 + c_2 = -3,$$

i.e. $c_2 = -2$. It follows that the solution of the IVP is

$$y(t) = e^{-t} - 2te^{-t} = (1 - 2t)e^{-t}.$$

4.2.20 The auxiliary equation $r^2 - 4r + 4 = 0$ has a double solution $r_1 = r_2 = 2$. It follows that the general solution of the equation $y'' - 4y' + 4y = 0$ is given by

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

The condition $y(1) = 1$ translates into $(c_1 + c_2)e^2 = 1$, while the condition $y'(1) = 1$ yields

$$(2c_1 + 3c_2)e^2 = 1,$$

i.e. $c_1 = 2e^{-2}$ and $c_2 = -e^{-2}$. It follows that the solution of the IVP is

$$y(t) = 2e^{2t-2} - te^{2t-2} = (2 - t)e^{2t-2}.$$

4.2.28 If y_1, y_2 were linearly dependent, then either y_2 would be 0, or we would have $y_1 = cy_2$ for some scalar c. Since $y_2 \neq 0$, we must have $y_1 = cy_2$, hence $e^{3t} = ce^{-4t} \Leftrightarrow e^{7t} = c$ for $t \in (0, 1)$. This is of course impossible, since the function e^{7t} is not constant, so the functions are linearly independent.

4.2.35 (a) On the interval $(0, \infty)$ we have $y_1 - y_2 = 0$, so the functions are linearly dependent.

(b) On the interval $(-\infty, 0)$ we have $y_1 + y_2 = 0$, so the functions are linearly dependent.

(c) If y_1, y_2 were dependent, there would exist a scalar c such that $y_1 = cy_2$ (because $y_2 \neq 0$). The equality from (a) says that $c = 1$, while the one from part (b) says that $c = -1$, which is impossible. This shows that y_1, y_2 are linearly independent on $(-\infty, \infty)$.

(d) The problem is that y_1, y_2 is not a pair of solutions to a second order homogeneous linear differential equation $a(t)y'' + b(t)y' + c(t)y = 0$. If they were, then since $y_1(1) = y_2(1) = 1$ and $y_1'(1) = y_2'(1) = 3$, they would provide two distinct solutions to the initial value problem

$$\begin{cases} a(t)y'' + b(t)y' + c(t)y = 0 \\ y(1) = 1, \quad y'(1) = 3 \end{cases}.$$

This is however impossible, since we know that the solution to an initial value problem is always unique.

4.3.3 The auxiliary equation $r^2 - 6r + 10 = 0$ has complex conjugated solutions $r_1 = 3 + i$ and $r_2 = 3 - i$. It follows that the general solution of the equation $y'' - 6y' + 10y = 0$ is given by

$$y(t) = c_1 e^{3t} \cos(t) + c_2 e^{3t} \sin(t).$$

4.3.16 The auxiliary equation $r^2 - 3r - 11 = 0$ has distinct solutions $r_1 = \frac{3+\sqrt{53}}{2}$ and $r_2 = \frac{3-\sqrt{53}}{2}$. It follows that the general solution of the equation $y'' - 3y' - 11y = 0$ is given by

$$y(t) = c_1 e^{\frac{3+\sqrt{53}}{2}t} + c_2 e^{\frac{3-\sqrt{53}}{2}t}.$$

4.3.25 The auxiliary equation $r^2 - 2r + 2 = 0$ has complex conjugated solutions $r_1 = 1 + i$ and $r_2 = 1 - i$. It follows that the general solution of the equation $y'' - 2y' + 2y = 0$ is given by

$$y(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t).$$

The condition $y(\pi) = e^\pi$ translates into $c_1 = -1$, while the condition $y'(\pi) = 0$ yields

$$-c_1 e^\pi + c_2 e^\pi = 0,$$

i.e. $c_2 = -c_1 = 1$. It follows that the solution of the IVP is

$$y(t) = -e^t \cos(t) + e^t \sin(t).$$

4.3.29 (a) The auxiliary equation $r^3 - r^2 + r + 3 = 0$ has solutions $r_1 = -1$, $r_2 = 1 + i\sqrt{2}$ and $r_3 = 1 - i\sqrt{2}$. It follows that the general solution of the equation $y''' - y'' + y' + 3y = 0$ is given by

$$y(t) = c_1 e^{-t} + c_2 e^t \cos(\sqrt{2}t) + c_3 e^t \sin(\sqrt{2}t).$$

(b) The auxiliary equation $r^3 + 2r^2 + 5r - 26 = 0$ has solutions $r_1 = 2$, $r_2 = -2 + 3i$ and $r_3 = -2 - 3i$. It follows that the general solution of the equation $y''' + 2y'' + 5y' - 26y = 0$ is given by

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} \cos(3t) + c_3 e^{-2t} \sin(3t).$$

(c) The auxiliary equation $r^4 + 13r^2 + 36 = 0$ is equivalent to $(r^2 + 4)(r^2 + 9) = 0$, so it has solutions $r_1 = 2i$, $r_2 = -2i$, $r_3 = 3i$ and $r_4 = -3i$. It follows that the general solution of the equation $y^{iv} + 13y'' + 36y = 0$ is given by

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos(3t) + c_4 \sin(3t).$$

4.3.35 For the door *not* to swing back and forth when closing, it must be that the solution function θ to the IVP

$$I\theta'' + b\theta' + k\theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = v_0,$$

does not alternate between positive and negative values. This can only happen if the formula for θ does not involve trigonometric functions, i.e. if the solutions of the auxiliary equation $Ir^2 + br + k = 0$ are not imaginary. According to the quadratic formula, this is the case when the discriminant $b^2 - 4Ik$ is nonnegative. Since $I, b, k > 0$, we get $b \geq 2\sqrt{Ik}$.