Review Session - Solutions

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1. We solve a) and b) simultaneously by augmenting A with the corresponding vectors (we let $b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$):

$$[A|b_1|b_2] = \begin{bmatrix} a & 1 & 1 & 0 & 1 & 1 \\ 0 & a & 1 & 1 & 1 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \underset{R_2 \leftrightarrow R_4}{\overset{R_1 \leftrightarrow R_3}{\sim}} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ a & 1 & 1 & 0 & 1 & 1 \\ 0 & a & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\overset{R_3=R_3-aR_1}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1+a & 0 & 1 & 1 \\ 0 & a & 1 & 1 & 1 & 2 \end{bmatrix} \overset{R_4=R_4-aR_2}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1+a & 1 & 1 & 1 \\ 0 & 0 & 1 & 1+a & 1 & 2 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1+a & 1 & 2 \\ 0 & 0 & 1+a & 1 & 1 & 1 \end{bmatrix} \overset{R_4=R_4-(1+a)R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1+a & 1 & 2 \\ 0 & 0 & 0 & -2a-a^2 & -a & -1-a \end{bmatrix}.$$

Now if $-2a-a^2 \neq 0$, then $-2a-a^2$ is a pivot, hence neither of b_1, b_2 is a pivot column, i.e. both systems are consistent.

Suppose now that $-2a-a^2 = 0$, i.e. $a = 0$ or $a = -2$. If $a = 0$, the matrix $[A|b_1]$ is in echelon form, and doesn't have a pivot in the b_1-column, hence the system $Ax = b_1$ is consistent. However, the matrix $[A|b_2]$ has a pivot in the b_2-column ($-1-a = -1 \neq 0$), hence $Ax = b_2$ is not consistent.

If $a = -2$, both $[A|b_1]$ and $[A|b_2]$ have a pivot in the last column, hence neither of the two systems are consistent.

2. (a) The matrix of T has columns $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $T(t) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $T(t^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, so the matrix of T is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) We row reduce A to find its null space:

$$A \overset{\substack{R_2=R_2-R_1\\R_3=R_3-R_1}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{R_4=R_4+R_2/2}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since there is a pivot in every column, $\mathrm{Nul}(A) = 0$, hence T is one-to-one. It follows that $\mathrm{Ker}(T) = 0$ with basis the empty set, and the image of T is 3-dimensional with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

3. The characteristic polynomial $\det(A - \lambda I) = -\lambda^3 + 2\lambda^2$, so A has eigenvalues $\lambda_1 = 2$ with multiplicity 1, and $\lambda_{2,3} = 0$ with multiplicity 2. A matrix is diagonalizable if and only if for each eigenvalue λ, the multiplicity of λ equals the dimension of the eigenspace E_λ. E_λ is always at least one-dimensional, so for λ of multiplicity one this condition is trivial. It follows that in our case A is diagonalizable if and only if the dimension of $E_0 = \mathrm{Nul}(A - 0 \cdot I) = \mathrm{Nul}(A)$ equals 2, the multiplicity of the eigenvalue 0.

   To determine $\mathrm{Nul}(A)$, we row reduce A:

$$A \overset{\substack{R_2=R_2-R_1/2\\R_3=R_3-R_1}}{\sim} \begin{bmatrix} 4 & -1 & -2 \\ 0 & 1/2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{R_3=R_3+2R_2}{\sim} \begin{bmatrix} 4 & -1 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   It follows that A has two pivot columns and one non-pivot column. This shows that the dimension of $\mathrm{Nul}(A)$ is 1, hence A is not diagonalizable.

4. We first solve the homogeneous equation $y'' + 3y' + 2y = 0$: the auxiliary equation is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$. It follows that

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

   We now use superposition to find a particular solution $y_p = y_{p_1} + y_{p_2}$, with y_{p_1} a solution for

$$y'' + 3y' + 2y = \sin(e^x),$$

   and y_{p_2} a solution for

$$y'' + 3y' + 2y = -e^{-x}.$$

   We use variation of parameters to determine y_{p_1}. We take

$$y_{p_1} = u_1 y_1 + u_2 y_2$$

   where $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and u_1', u_2' satisfy the system of equations

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \sin(e^x) \end{cases}$$

or equivalently

$$\begin{cases} u_1'e^{-x} + u_2'e^{-2x} = 0 \\ u_1'(-e^{-x}) + u_2'(-2e^{-2x}) = \sin(e^x) \end{cases}$$

Adding the two equations we obtain

$$-u_2'e^{-2x} = \sin(e^x),$$

or equivalently

$$u_2' = -e^{2x}\sin(e^x).$$

It follows that

$$u_2 = \int(-e^{2x}\sin(e^x))dx = e^x\cos(e^x) - \sin(e^x)$$

(this follows from the substitution $u = e^x$). We have

$$u_1' = -u_2'e^{-x} = e^x\sin(e^x)$$

so

$$u_1 = \int(e^x\sin(e^x))dx = -\cos(e^x)$$

(again by the substitution $u = e^x$). Substituting back into the formula of y_{p_1} we get

$$y_{p_1} = -\cos(e^x)e^{-x} + (e^x\cos(e^x) - \sin(e^x))e^{-2x} = -\sin(e^x)e^{-2x}$$

We now use undetermined coefficients to find y_{p_2}. Since -1 is a solution of the characteristic equation, with multiplicity $s = 1$, and e^{-x} has the form *(exponential)·(polynomial of degree 0)*, y_{p_2} has to equal x^s·*(exponential)·(polynomial of degree 0)*:

$$y_{p_2} = x \cdot e^{-x} \cdot a.$$

We get $y_{p_2}' = ae^{-x}(1-x)$, $y_{p_2}'' = ae^{-x}(x-2)$, and

$$-e^{-x} = y_{p_2}'' + 3y_{p_2}' + 2y_{p_2} = ae^{-x}(x - 2 + 3(1-x) + 2x) = ae^{-x}.$$

It follows that $a = -1$ and $y_{p_2} = -xe^{-x}$. We get

$$y_p = y_{p_1} + y_{p_2} = -\sin(e^x)e^{-2x} - xe^{-x}.$$

Putting everything together, it follows that the general solution of our equation is given by

$$y = c_1e^{-x} + c_2e^{-2x} - \sin(e^x)e^{-2x} - xe^{-x}.$$

5. We first solve the homogeneous equation $y^{(4)} + 2y^{(2)} + y = 0$: the auxiliary equation is $(r^2 + 1)^2 = 0$, with complex conjugate roots $\pm i$, each having multiplicity 2. It follows that a basis for the space of solutions of this homogeneous equation is $\{\cos(x), \sin(x), x\cos(x), x\sin(x)\}$, i.e.

$$y_h = c_1\cos(x) + c_2\sin(x) + c_3x\cos(x) + c_4x\sin(x).$$

We now use undetermined coefficients to find a particular solution y_p of $y^{(4)} + 2y^{(2)} + y = x\cos(x)$. Since the right hand side has the form $e^{\alpha x}\cos(\beta x)$·*polynomial of degree* 1, with

$\alpha = 0$, $\beta = 1$, we take y_p of the form $x^s \cdot e^{\alpha x} \cos(\beta x) \cdot$*polynomial of degree* 1, where s is the multiplicity of $\alpha + \beta i = i$ as a root of the auxiliary equation, i.e. $s = 2$. We get

$$y_p = x^2((a + bx)\cos(x) + (c + dx)\sin(x)) = (ax^2 + bx^3)\cos(x) + (cx^2 + dx^3)\sin(x).$$

Taking derivatives, we obtain

$$\begin{aligned} y_p' =& (2ax + (3b + c)x^2 + dx^3)\cos(x) \\ &+ (2cx + (3d - a)x^2 - bx^3)\sin(x), \end{aligned}$$

$$\begin{aligned} y_p^{(2)} =& (2a + (6b + 4c)x + (6d - a)x^2 - bx^3)\cos(x) \\ &+ (2c + (6d - 4a)x - (6b + c)x^2 - dx^3)\sin(x), \\ y_p^{(3)} =& ((6b + 6c) + (18d - 6a)x - (9b + c)x^2 - dx^3)\cos(x) \\ &+ ((6d - 6a) - (18b + 6c)x - (9d - a)x^2 + bx^3)\sin(x), \\ y_p^{(4)} =& ((24d - 12a) - (36b + 8c)x - (12d - a)x^2 + bx^3)\cos(x) \\ &+ (-(24b + 12c) - (36d - 8a)x + (12b + c)x^2 + dx^3)\sin(x). \end{aligned}$$

It follows that

$$x\cos(x) = y_p^{(4)} + 2y_p^{(2)} + y_p = ((24d - 8a) - 24bx)\cos(x) + (-(24b + 8c) - 24dx)\sin(x).$$

Equating the coefficients, we obtain $24d - 8a = 0$, $-24b = 1$, $24b + 8c = 0$ and $24d = 0$. This shows that $a = d = 0$ and $b = -1/24, c = 1/8$. In conclusion,

$$y_p = -\frac{x^3\cos(x)}{24} + \frac{x^2\sin(x)}{8},$$

and the general solution of our equation is given by

$$y = c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x) - \frac{x^3\cos(x)}{24} + \frac{x^2\sin(x)}{8}.$$