Review Session

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1. Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$, defined by

   $T(x) = (-3x_1 + 6x_2 - x_3 + x_4 - 7x_5, x_1 - 2x_2 + 2x_3 + 3x_4 - x_5, 2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5)$.

   a) Find the standard matrix of T.

   b) Find bases for $\text{Nul}(A)$, $\text{Col}(A)$, $\text{Row}(A)$.

   c) Find bases for $\text{Nul}(A^T)$, $\text{Col}(A^T)$, $\text{Row}(A^T)$.

   d) Find the ranks of A, A^T, A^TA and AA^T.

   e) Find bases for the kernel of T and for the range of T.

   f) Is T one-to-one? Is it onto?

2. Consider the bases $\mathcal{B} = \{1 + t + t^2, 1 + t - 2t^2, 1 - t\}$, $\mathcal{C} = \{1, (t-1), (t-1)^2\}$, and $\mathcal{D} = \{1, (t-1), (t-1)t\}$ for $\mathbb{P}_2$. Find the representation of the polynomial $p = (t+1)^2$ in each of these three bases. Compute the change of coordinates matrix from $\mathcal{D}$ to $\mathcal{B}$.

3. Consider an $n \times n$ invertible matrix A. Is A row equivalent to A^{-1}? Show that if A is similar to A^{-1}, then $\det(A) = \pm 1$. Give an example of a matrix A with determinant 1 which is not similar to its inverse.

4. Consider the matrix
$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
   Find the eigenvalues of A. Determine the values of h for which A is diagonalizable.

5. Consider the sequence given by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n - f_{n-1}$ for $n \geq 1$. Show that
$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix},$$
   for all $n \geq 1$. Find f_{1000}.

6. Find an orthogonal basis for $W = \text{Span}\{u_1, u_2, u_3\}$, where $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$. Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(x) = \text{proj}_W(x)$. Show

that T is a linear transformation, determine $T(v)$, where $v = \begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \end{bmatrix}$, and compute the standard matrix of T. Is T one-to-one? Describe the range of T.

7. Describe all least-squares solutions of the equation $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}.$$

8. Calculate the characteristic equation of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

9. Let A be an $m \times n$ matrix. Show that $\text{Nul}(A) = \text{Nul}(A^T A)$, and that $\text{Col}(A^T) = \text{Col}(A^T A)$.