Worksheet 11

Claudiu Raicu

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(1-3) Find a general solution for the differential equation

1. $y''' + 3y'' - 4y' - 6y = 0$.
2. $y^{(4)} + 4y'' + 4y = 0$.
3. $(D-1)^2(D+3)(D^2+2D+5)^2[y] = 0$, where D is the differentiation operator $\dfrac{\partial}{\partial x}$.
4. Show that if a, b, c are distinct real numbers, then the functions e^{ax}, e^{bx}, e^{cx} are linearly independent on the interval $(-\infty, \infty)$. Deduce that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \neq 0.$$

(5-10) Express the given system of differential equations in matrix notation, or the given higher-order differential equation as a matrix system in normal form.

5. $\begin{cases} x_1' = x_1 - x_2 + x_3 - x_4 \\ x_2' = x_1 + x_4 \\ x_3' = \sqrt{\pi} x_1 - x_3 \\ x_4' = 0 \end{cases}$

6. $\begin{cases} x'' - 3x' + t^2 y - \cos(t) x = 0 \\ y''' + y'' - tx' + y' + e^t x = 0 \end{cases}$

7. The damped mass-spring oscillator equation $my'' + by' + ky = 0$.

8. Bessel's equation $y'' + \dfrac{1}{t} y' + \left(1 - \dfrac{n^2}{t^2}\right) y = 0$.

9. $\begin{cases} r'(t) = 2r(t) + \sin(t) \\ \theta'(t) = r(t) - \theta(t) + 1 \end{cases}$

10. $y''(t) - 3y'(t) - 10y(t) = \sin(t)$.

11. Verify that the vector functions $x_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $x_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$ are solutions to the homogeneous system

$$x' = Ax = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$$

on $(-\infty, \infty)$, and that

$$x_p = \frac{3}{2}\begin{bmatrix} te^t \\ te^t \end{bmatrix} - \frac{1}{4}\begin{bmatrix} e^t \\ 3e^t \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a particular solution to the nonhomogeneous system $x' = Ax + f(t)$, where $f(t) = \operatorname{col}(e^t, t)$. Find a general solution to $x' = Ax + f(t)$.

(12-13) Find a general solution of the system $x'(t) = Ax(t)$ for the given matrix A. Write a fundamental matrix for this system.

12. $A = \begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix}$.

13. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix}$.

14. Solve the initial value problem

$$x'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot x(t), \quad x(0) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

15. Use the substitution $x_1 = y$, $x_2 = y'$ to convert the linear equation $ay'' + by' + cy = 0$, where a, b and c are constants, into a normal system. Show that the characteristic equation for this system is the same as the auxiliary equation for the original equation.