Worksheet 13 - Solutions

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1. The auxiliary equation for $y'' + y = 0$ is $r^2 + 1 = 0$, with solutions $\pm i$. It follows that the general solution of $y'' + y = 0$ is given by
$$y = c_1 \cos(x) + c_2 \sin(x).$$
a) $y(0) = 0$ implies $c_1 = 0$. $y(2\pi) = 1$ implies $c_1 = 1$, i.e. there is no y satisfying the given boundary conditions.

b) $y(0) = 1$ implies $c_1 = 1$, and $y(2\pi) = 1$ also implies $c_1 = 1$. It follows that $c_1 = 1$ and c_2 is arbitrary, i.e. $y = c\sin(x)$ for some real number c.

2. The auxiliary equation for $y'' + \lambda y = 0$ is $r^2 + \lambda = 0$. We analyze the possible cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$, corresponding to the auxiliary equation having a double root, distinct real roots, or complex conjugate roots.

<u>*Case* $\lambda = 0$.</u> We have that $r^2 = 0$ has a double root $r = 0$. This means that $y = c_1 + c_2 x$ for some c_1, c_2. We analyze a) and b) separately:

a) $y(0) = 0$ yields $c_1 = 0$, while $y'(\pi) = 0$ yields $c_2 = 0$, i.e. $y = 0$ is the unique solution in this case.

b) $y(0) - y'(0) = 0$ yields $c_1 - c_2 = 0$, and $y(\pi) = 0$ yields $c_1 + c_2\pi = 0$. This means that $c_1 = c_2$ and $(1 + \pi)c_2 = 0$, i.e. $c_1 = c_2 = 0$. Therefore $y = 0$ is the unique solution in this case also.

<u>*Case* $\lambda < 0$.</u> We have that $r^2 = -\lambda$ has distinct real roots $r_{1,2} = \pm\sqrt{-\lambda}$. This means that $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ for some c_1, c_2. We analyze a) and b) separately:

a) $y(0) = 0$ yields $c_1 + c_2 = 0$, while $y'(\pi) = 0$ yields
$$c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - c_2\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} = 0.$$
Dividing by $\sqrt{-\lambda}$ and substituting c_2 by $-c_1$, we get
$$c_1(e^{\sqrt{-\lambda}\pi} + e^{-\sqrt{-\lambda}\pi}) = 0,$$
yielding $c_1 = 0$, and hence $c_2 = 0$. We get that $y = 0$ is the unique solution in this case.

b) $y(0)-y'(0) = 0$ yields $c_1+c_2-(c_1\sqrt{-\lambda}-c_2\sqrt{-\lambda}) = 0$, i.e. $c_1(1-\sqrt{-\lambda})+c_2(1+\sqrt{-\lambda}) = 0$. $y(\pi) = 0$ yields
$$c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi} = 0.$$
This means c_1, c_2 satisfies a homogeneous system of equations whose coefficient matrix is
$$A = \begin{bmatrix} 1 - \sqrt{-\lambda} & 1 + \sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{bmatrix}$$

This is an invertible matrix, since its determinant is nonzero:

$$\det(A) = (1-\sqrt{-\lambda})e^{-\sqrt{-\lambda}\pi} - (1+\sqrt{-\lambda})e^{\sqrt{-\lambda}\pi}.$$

If this was zero, multiplying by $e^{\sqrt{-\lambda}\pi}$ we'd get

$$1-\sqrt{-\lambda} = (1+\sqrt{-\lambda})e^{2\sqrt{-\lambda}\pi}.$$

This is impossible, since the LHS is smaller than 1, whereas the RHS is larger than 1.

Now because A is invertible, its null space is zero, i.e. the vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ which is in Nul(A) has to be the zero vector. We get $c_1 = c_2 = 0$, and hence $y = 0$ is the unique solution of the problem.

Case $\lambda > 0$. We have that $r^2 = -\lambda$ has complex conjugate roots $r_{1,2} = \pm i\sqrt{\lambda}$. This means that $y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ for some c_1, c_2. We analyze a) and b) separately:

a) $y(0) = 0$ yields $c_1 = 0$, so $y(x) = c_2\sin(\sqrt{\lambda}x)$. $y'(\pi) = 0$ yields $c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$. If $\cos(\sqrt{\lambda}\pi) \neq 0$, then c_2 must be equal to zero, and therefore $y = 0$.

In order to get a nontrivial solution y, we must have $\cos(\sqrt{\lambda}\pi) = 0$. We know that $\cos(\alpha\pi) = 0$ if and only if $\alpha = n + \frac{1}{2}$ for some integer n. Since $\sqrt{\lambda}$ is positive, we must have $\sqrt{\lambda} = n + \frac{1}{2}$ for some nonnegative integer n. This implies that for

$$\lambda = (n+\frac{1}{2})^2,\ n = 0, 1, 2, \cdots$$

the problem has a nontrivial solution, and the set of solutions is given by

$$y_n = c\sin((n+\frac{1}{2})x).$$

b) $y(0) - y'(0) = 0$ yields $c_1 - c_2\sqrt{\lambda} = 0$. The condition $y(\pi) = 0$ yields

$$c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) = 0.$$

Using that $c_1 = c_2\sqrt{\lambda}$, this gives

$$c_2(\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi)) = 0.$$

If $\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) \neq 0$, then $c_2 = 0$ and hence $c_1 = 0$, i.e. $y = 0$.

In order to get a nontrivial solution y, we must have $\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) = 0$, i.e.

$$\sqrt{\lambda} + \tan(\sqrt{\lambda}\pi) = 0.$$

Unfortunately, there are no formulas for the λ's that satisfy this equation. The first few λ's are given by $1.29, 2.37, 3.41, \cdots$.

3. We look for a solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta n^2 t}\sin\left(\frac{n\pi x}{L}\right),$$

where $\beta = 3$ and $L = \pi$, i.e.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx).$$

Letting $t = 0$, we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx).$$

a) $u(x,0) = f(x)$ yields $c_1 = 1$, $c_4 = -6$ and $c_n = 0$ for $n \neq 1, 4$. This shows that

$$u(x,t) = e^{-3t} \sin(x) - 6e^{-48t} \sin(4x).$$

b) $u(x,0) = f(x)$ yields $c_1 = 1$, $c_3 = -7$, $c_5 = 1$ and $c_n = 0$ for $n \neq 1, 3, 5$. This shows that

$$u(x,t) = e^{-3t} \sin(x) - 7e^{-27t} \sin(3x) + e^{-75t} \sin(5x).$$

4. We look for a solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi\alpha}{L} t\right) + b_n \sin\left(\frac{n\pi\alpha}{L} t\right) \right) \sin\left(\frac{n\pi x}{L}\right),$$

where $L = \pi$ and $\alpha = 3$, i.e.

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(3nt) + b_n \sin(3nt)) \sin(nx).$$

We get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = f(x),$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 3nb_n \sin(nx) = g(x).$$

a) Since $g(x) \equiv 0$, we get $b_n = 0$ for all n. The condition $u(x,0) = f(x)$ yields $a_2 = 3$, $a_{13} = 12$ and $a_n = 0$ for $n \neq 2, 13$. It follows that

$$u(x,t) = 3\cos(6t)\sin(2x) + 12\cos(39t)\sin(13x).$$

b) The condition $u(x,0) = f(x)$ yields $a_2 = 6$, $a_6 = 2$ and $a_n = 0$ for $n \neq 2, 6$. The condition $\frac{\partial u}{\partial t}(x,0) = g(x)$ yields $27b_9 = 11$, $45b_{15} = -14$ and $b_n = 0$ for $n \neq 9, 15$. This gives $b_9 = 11/27$, $b_{15} = -14/45$ and therefore

$$u(x,t) = 6\cos(6t)\sin(2x) + 2\cos(18t)\sin(6x) + \frac{11}{27}\sin(27t)\sin(9x) - \frac{14}{45}\sin(45t)\sin(15x).$$

5. The coefficients in the Fourier series are given by the formulas

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx,$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{n\pi x}{T}\right)dx.$$

a) Since $T=\pi$, and f is even, we get $b_n = 0$ for all n, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2((-1)^n-1)}{\pi n^2}.$$

b) Since $T=2$, $f(x)=1$ for $x<0$ and $f(x)=x$ for $x>0$, we have

$$a_0 = \frac{1}{2}\left(\int_{-2}^{0} 1dx + \int_0^2 xdx\right) = 2,$$

$$a_n = \frac{1}{2}\left(\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx\right) = \frac{2((-1)^n-1)}{\pi^2 n^2},\quad n\geq 1$$

and

$$b_n = \frac{1}{2}\left(\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx\right) = \frac{1}{2}\left(\frac{2((-1)^n-1)}{\pi n} + \frac{4(-1)^{n+1}}{\pi n}\right)$$

$$= \frac{(-1)^{n+1}-1}{\pi n}.$$