Worksheet 4

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September 17, 2010

1. Let W be the union of the first and third quadrants in the xy-plane.

   a) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

   b) Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u}+\mathbf{v}$ is not in W. Is W a vector space?

2. Consider the line $x+y=1$ in $\mathbb{R}^2$. Is it a vector space?

3. Determine if the given set is a subspace of $\mathbb{P}_n$ (polynomials of degree at most n) for an appropriate value of n.

   a) All polynomials of the form $p(t)=at^2$, where a is in $\mathbb{R}$.

   b) All polynomials of the form $p(t)=a+t^2$, where a is in $\mathbb{R}$.

   c) All polynomials of degree at most 3, with integers as coefficients.

   d) All polynomials of degree 4.

   e) All polynomials f of degree at most 5 with the property that $4f(x)=xf'(x)$.

4. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA=0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

5. Given subspaces H and K of a vector space V, the sum of H and K, written as $H+K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H+K=\{\mathbf{w}:\mathbf{w}=\mathbf{u}+\mathbf{v} \text{ for some } \mathbf{u} \text{ in } H \text{ and some } \mathbf{v} \text{ in } K.\}.$$

   a) Show that $H+K$ is a subspace of V.

   b) Show that H is a subspace of $H+K$ and K is a subspace of $H+K$.

6. Find the null spaces of the matrices

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

7. Define a linear transformation $T:\mathbb{P}_2\to\mathbb{R}^2$ by

$$T(\mathbf{p})=\begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$$

   Find polynomials $\mathbf{p}_1,\mathbf{p}_2$ that span the kernel of T, and describe the range of T.

8. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$.

   a) Show that T is a linear transformation.

   b) Let B be any element of $M_{2\times 2}$ such that $B^T = B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

   c) Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.

   d) Describe the kernel of T.

9. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

10. Find bases for the null spaces of the matrices in exercise 6.

11. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$. Verify that $4\mathbf{v}_1 + 5\mathbf{v}_2 = 3\mathbf{v}_3$ and use this information to find a basis for $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

12. Suppose $\mathbb{R}^4 = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

13. Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$ with $k < n$. Explain why S cannot be a basis for $\mathbb{R}^n$.

14. Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_2$? Is it a basis? Why or why not?

15. Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 1 - t$ and $\mathbf{p}_3(t) = 2$. Write a linear dependence relation among $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. Find a basis for $\mathrm{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.