Worksheet 5

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1. Find the coordinate vector $[x]_{\mathcal{B}}$ of the vector x relative to the given basis $\mathcal{B} = \{b_1, \cdots, b_n\}$:

   a) $b_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

   b) $b_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $x = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$.

2. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$, relative to $\mathcal{B}$.

3. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

4. Given a vector space V with basis $\mathcal{B}$, show that the coordinate mapping $x \mapsto [x]_{\mathcal{B}}$ is one-to-one and onto.

5. Use coordinate vectors to test the linear independence of the sets of polynomials

   a) $1 - 2t^2 - 3t^3$, $t + t^3$, $1 + 3t - 2t^2$.

   b) $(t-1)^2$, $t^3 - 2$, $(t-2)^3$.

6. Determine the dimensions of $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$ (the row space of A) for the matrix
$$\begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

8. Let $T : V \to W$ be a linear transformation between vector spaces V, W, let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vector in H. Show that $T(H)$ is a subspace of W and that $\dim(T(H)) \leq \dim(H)$. Explain why equality holds when T is one-to-one.

9. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?

10. Find bases for and compute the dimensions of Col(A), Row(A) and Nul(A), where

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

What is the rank of A?

11. If a 3×8 matrix A has rank 3, find the dimensions of Nul(A), Row(A) and rankA^T.

12. Suppose A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$. What has to be true about the two numbers rank$[A\ b]$ and rank A for the equation $Ax = b$ to be consistent?

13. Calculate the rank of the matrix uv^T, where $u = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$. What happens if you replace v by a nonzero vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$?

14. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ be bases for a vector space V, and suppose that $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$ and $f_3 = -3d_1 + 2d_3$.

    a) Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

    b) Find $[x]_{\mathcal{D}}$ for $x = f_1 - 2f_2 + 2f_3$.

15. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.