Worksheet 5 - Solutions

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1. a) We have

$$[b_1\ b_2] \cdot [x]_{\mathcal{B}} = x,$$

so to find $[x]_{\mathcal{B}}$ we need to solve the system whose augmented matrix is

$$[b_1\ b_2\ x] = \begin{bmatrix} 1 & 5 & 4 \\ -2 & -6 & 0 \end{bmatrix}.$$

Row-reducing, we obtain the reduced echelon form of $[b_1\ b_2\ x]$ to be

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \end{bmatrix},$$

so

$$[x]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}.$$

b) We have

$$[b_1\ b_2\ b_3] \cdot [x]_{\mathcal{B}} = x,$$

so to find $[x]_{\mathcal{B}}$ we need to solve the system whose augmented matrix is

$$[b_1\ b_2\ b_3\ x] = \begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix}.$$

Row-reducing, we obtain the reduced echelon form of $[b_1\ b_2\ b_3\ x]$ to be

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

so

$$[x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

2. We must find c_1, c_2, c_3 such that

$$c_1(1-t^2) + c_2(t-t^2) + c_3(2-2t+t^2) = 3+t-6t^2.$$

Equating for the coefficients we obtain a system of equations whose augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix}.$$

Row-reducing, we obtain the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

so

$$[p]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

3. We are trying to solve the system of equations

$$x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This has augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{bmatrix},$$

whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

x_3 is then a free variable and the solutions to this system are given by $x_1 = 5 + 5x_3$, $x_2 = -2 - x_3$ and x_3 arbitrary. To get two distinct solutions, we choose two distinct values for x_3.

If $x_3 = 0$, we get $x_1 = 5$ and $x_2 = -2$, so

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5v_1 - 2v_2.$$

If $x_3 = -1$, we get $x_1 = 0$ and $x_2 = -1$, so

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -v_2 - v_3.$$

4. Say $\mathcal{B} = \{b_1, \cdots, b_n\}$, and consider arbitrary vectors $x, y \in V$ and scalar $c \in \mathbb{R}$. We denote by T the linear transformation given by $T(x) = [x]_{\mathcal{B}}$. To show that T is linear we need to check that

a) $T(x + y) = T(x) + T(y)$.

b) $T(c \cdot x) = c \cdot T(x)$.

Let's write

$$x = x_1 \cdot b_1 + \cdots + x_n \cdot b_n,$$

and

$$y = y_1 \cdot b_1 + \cdots + y_n \cdot b_n,$$

for some real numbers $x_1, \cdots, x_n$, $y_1, \cdots, y_n$. This means that

$$T(x) = [x]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } T(y) = [y]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

To see why a) is true, note that

$$x + y = (x_1 + y_1) \cdot b_1 + \cdots + (x_n + y_n) \cdot b_n,$$

i.e.

$$T(x+y) = [x+y]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = T(x) + T(y).$$

To see why b) is true, note that

$$c \cdot x = (c \cdot x_1) \cdot b_1 + \cdots + (c \cdot x_n) \cdot b_n,$$

i.e.

$$T(c \cdot x) = [c \cdot x]_{\mathcal{B}} = \begin{bmatrix} c \cdot x_1 \\ c \cdot x_2 \\ \vdots \\ c \cdot x_n \end{bmatrix} = c \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c \cdot T(x).$$

5. If we write $\mathcal{C}$ for the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$, then we can use the coordinate mapping $p \mapsto [p]_{\mathcal{C}}$ to represent all our polynomials as vectors in $\mathbb{R}^4$.

   a) The coordinate mapping sends the polynomials to the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}.$$

   Row-reducing, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   i.e. the matrix doesn't have a pivot in every column, so its columns are linearly dependent. It follows that our original vectors were linearly dependent.

   b) The coordinate mapping sends the polynomials to the columns of the matrix

$$\begin{bmatrix} 1 & -2 & -8 \\ -2 & 0 & 12 \\ 1 & 0 & -6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e. the matrix doesn't have a pivot in every column, so its columns are linearly dependent. It follows that our original vectors were linearly dependent.

6. The matrix is already in echelon form, and we circle its pivots:

$$\begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since A has 3 pivots, its columns space has dimension 3. The dimension of the row space of A is the same as the rank of A which is also the dimension of the columns space of A, and is therefore equal to 3. The dimension of the null space of A is the number of non-pivot columns, i.e. $5 - 3 = 2$.

7. Consider a basis $\mathcal{B}$ of H. Since H is n-dimensional, $\mathcal{B}$ has size n. $\mathcal{B}$ is a subset of H, which in turn is a subset of V, so $\mathcal{B}$ consists of n linearly independent vectors in V. But since V also has dimension n, the Basis Theorem says that $\mathcal{B}$ is automatically a basis for V.

8. We first show that $T(H)$ is a subspace of W. In order to do that, we must check that it is closed under addition and scalar multiplication. Consider two elements $T(h_1)$ and $T(h_2)$ in $T(H)$. Their sum is $T(h_1) + T(h_2)$, which by the linearity property of T, is the same as $T(h_1 + h_2) \in T(H)$, so $T(H)$ is closed under addition. Now consider an element $T(h) \in T(H)$ and a scalar c. We have $cT(h) = T(ch) \in T(H)$, so $T(H)$ is also closed under scalar multiplication, and is therefore a subspace of W.

   Now to show that $\dim(T(H)) \leq \dim(H)$, we consider a basis $\mathcal{B} = \{b_1, \cdots, b_n\}$ of H, where $n = \dim(H)$. Then any element of $T(H)$ can be written as a linear combination of $T(b_1), \cdots, T(b_n)$. To see that, take $h \in H$. Since $\mathcal{B}$ is a basis of H, we can write

$$h = c_1 \cdot b_1 + \cdots + c_n \cdot b_n.$$

   Applying T to both sides and using the linearity of T we obtain

$$T(h) = c_1 \cdot T(b_1) + \cdots + c_n \cdot T(b_n),$$

   which is certainly a linear combination of $T(b_1), \cdots, T(b_n)$.

   It follows that $T(b_1), \cdots, T(b_n)$ is a spanning set for $T(H)$ consisting of n elements. By the Spanning Set Theorem, a subset of $T(b_1), \cdots, T(b_n)$ is a basis for $T(H)$. This means that the dimension of $T(H)$ is at most n.

9. We have in general

$$\dim(\mathrm{Nul}(A)) + \mathrm{rank}(A) = n, \text{ the number of columns of A.}$$

   In our case $n = 9$, so

$$\mathrm{rank}(A) = 9 - 2 = 7.$$

10. The echelon form of A is

$$\begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since A has 3 pivot columns, the dimension of its column space is 3, which is the same as the dimension of its row space and the same as the rank of A. The dimension of the null space of A is the number of non-pivot columns, i.e. $5 - 3 = 2$.

11. The rank of A^T is the same as the rank of A, i.e. it is equal to 3. The dimension of the row space of A is also equal to the rank of A, i.e. it is 3. Since

$$\dim(\mathrm{Nul}(A)) + \mathrm{rank}(A) = n, \text{ the number of columns of A},$$

and $n = 8$ in our case, it follows that the dimension of the null space of A is $8 - 3 = 5$.

12. The equation $Ax = b$ is consistent if and only if the augmented matrix $[A\ b]$ doesn't have a pivot in the last column. This happens if and only if the matrices A and $[A\ b]$ have the same pivot columns, if and only if they have the same number of pivot columns. Since the number of pivot columns of a matrix equals its rank, this happens if and only if the rank of A equals the rank of $[A\ b]$. Summing up, we have that the equation $Ax = b$ is consistent if and only if the rank of A equals the rank of $[A\ b]$.

13. We have

$$uv^T = \begin{bmatrix} 2 & -2 & -6 \\ -3 & 3 & 9 \\ 5 & -5 & -15 \end{bmatrix}.$$

Notice that the 2nd and 3rd columns of this matrix are multiples of the first one, so the columns space of uv^t is generated by its first column, so the rank of uv^T is 1.

This holds much more generally: for any two nonzero column vectors $u, v \in \mathbb{R}^n$, the matrix uv^T has rank 1. Let's check this in the case when u is as above, and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an arbitrary nonzero vector. Then the columns of uv^T are just

$$a \cdot u, \ b \cdot u, \text{ and } c \cdot u.$$

Since not all a, b, c are equal to zero, it follows that the column space of uv^T is generated by the vector u, which is therefore a basis for $\mathrm{Col}(uv^T)$, hence the rank of uv^T is 1.

14. a) By definition,

$$\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \left([f_1]_{\mathcal{D}}, [f_2]_{\mathcal{D}}, [f_3]_{\mathcal{D}}\right).$$

We have

$$[f_1]_{\mathcal{D}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad [f_2]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [f_3]_{\mathcal{D}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix},$$

so

$$\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

b) $[x]_\mathcal{D}$ is given by

$$[x]_\mathcal{D} = \underset{\mathcal{D}\leftarrow\mathcal{F}}{P} \cdot [x]_\mathcal{F},$$

where

$$[x]_\mathcal{F} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

It follows that

$$[x]_\mathcal{D} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}.$$

15. Denote the basis elements of $\mathcal{B}$ by b_1, b_2, b_3, and the basis elements of $\mathcal{C}$ by c_1, c_2, c_3. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of b_1, b_2, b_3, namely

$$[b_1]_\mathcal{C} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad [b_2]_\mathcal{C} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \quad [b_3]_\mathcal{C} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

It follows that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}.$$

We would like to calculate the $\mathcal{B}$-coordinate vector $[p]_\mathcal{B}$ of $p = -1 + 2t$. We have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [p]_\mathcal{B} = [p]_\mathcal{C} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},$$

so in order to find $[p]_\mathcal{B}$ we need to solve the system of equations whose augmented matrix is

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix}.$$

Row-reducing, we obtain that

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

so

$$[p]_\mathcal{B} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

This means that $p = 5b_1 - 2b_2 + b_3$, or

$$-1 + 2t = 5 \cdot (1 - 2t + t^2) - 2 \cdot (3 - 5t + 4t^2) + 1 \cdot (2t + 3t^2).$$