Worksheet 6

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1. Is $\lambda = 4$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

2. Find the eigenvalues of the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$

3. Construct an example of a 2×2 matrix with only one (distinct) eigenvalue. Can you find an example where the eigenspace corresponding to the unique eigenvalue is one dimensional?

4. Consideran $n \times n$ matrix with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. What happens if the column sums are all equal to s?

5. Find the characteristic polynomial and the eigenvalues of the matrices

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix}.$$

6. Find the characteristic polynomial and the eigenvalues of the matrices

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}.$$

7. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

8. Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

9. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

10. Diagonalize the matrix

$$A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$$

and compute A^k as a function of k.

11. Diagonalize the following matrices, if possible.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}.$$

12. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

13. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) + t^2\mathbf{p}(t)$.

    a) Find the image of $\mathbf{p}(t) = 2 - t + t^2$.

    b) Show that T is a linear transformation.

    c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

    d) Find the kernel of T, and a basis for the image of T.

14. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

15. Verify the following statements:

    a) If A is invertible and similar to B, then B is invertible and A^{-1} is similar to B^{-1}.

    b) If A is similar to B, then A^2 is similar to B^2.

    c) If B is similar to A and C is similar to A, then B is similar to C.

    d) If $B = P^{-1}AP$ and x is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}x$ is an eigenvector of B corresponding also to λ.

    e) If A and B are similar, then they have the same rank.