Worksheet 6 - Solutions

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1. We first need to determine whether $A - 4I$ is invertible:

$$A - 4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so it is not invertible. It follows that $\lambda = 4$ is an eigenvalue of A. To find an eigenvector corresponding to the eigenvalue 4, we need to solve the system of equations

$$(A - 4I)x = 0.$$

We can read off the solutions to this system from the reduced echelon form of $A - 4I$ above. They are

$$\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases}$$

so an eigenvalue (which is also a basis for the eigenspace corresponding to the eigenvalue $\lambda = 4$) is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

2. A is upper-triangular, so its eigenvalues are the diagonal entries, namely $0, 2, -1$.

   B is lower-triangular, so its eigenvalues are the diagonal entries, namely $4, 0, -3$.

   (Convince yourselves that this is true!)

3. $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   This has only 0 as an eigenvalue. The corresponding eigenspace has the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a basis.

4. We have

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue s.

If the columns sums are equal to s, this means that the row sums of A^T are all s, so s is an eigenvalue for A^T. But A and A^T have the same eigenvalues, so s is still an eigenvalue for A.

5. $\det(A - \lambda I) = (\lambda - 3)^2$, so A has 3 as an eigenvalue with multiplicity 2.

   $\det(B - \lambda I) = \lambda^2 - 11\lambda + 40$, so the complex eigenvalues of B are given by

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{39}}{2}.$$

   B has no real eigenvalues!

6. Cofactor expansion along the 3rd row of A yields

$$\det(A - \lambda I) = (2 - \lambda)(-1 - \lambda)(4 - \lambda),$$

   so A has eigenvalues $2, -1$ and 4.

   Cofactor expansion along the 3rd column of B yields

$$\det(B - \lambda I) = (3 - \lambda)(\lambda - 5)(\lambda - 10),$$

   so B has eigenvalues $3, 5$ and 10.

7. Row-reducing the matrix $A - 5I$ we get

$$A - 5I \sim \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   The eigenspace for $\lambda = 5$ is precisely the null space of the matrix $A - 5I$. This is two dimensional if and only if $A - 5I$ has two nonpivot columns. The first column of $A - 5I$ is not a pivot, while the 2nd and 4th columns are pivot columns. It follows that $A - 5I$ has exactly two nonpivot columns if and only if its 3rd column is not a pivot column, if and only if $h - 6 = 0$, if and only if $h = 6$.

8. Since Q is invertible, it has an inverse Q^{-1}, so

$$Q^{-1}AQ = Q^{-1}(QR)Q = RQ = A_1,$$

   i.e. A and A_1 are similar.

9. Since the sum of the dimensions of the distinct eigenspaces corresponding to the eigenvalues of A is $3 + 2 = 5$, which is equal to the size of the matrix A, Theorem 7 says that A is diagonalizable.

10. The characteristic equation $\det(A - \lambda I) = 0$ for A is

$$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1),$$

with roots 2 and 1, so the eigenvalues of A are 2 and 1. One can check (do it!) that

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \text{ and } w = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

are eigenvalues for $\lambda = 2$ and $\lambda = 1$ respectively. Thus we can use the matrix $P = [v\ w]$ to diagonalize A, and we get that

$$P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

or equivalently,

$$A = PDP^{-1}.$$

Taking k-th powers, we obtain

$$A^k = (PDP^{-1})^k = PD^kP^{-1} = P \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 - 3 \cdot 2^k & 12 \cdot 2^k - 12 \\ 1 - 2^k & 4 \cdot 2^k - 3 \end{bmatrix}.$$

11. For the matrix A, the multiplicity of $\lambda = 4$ is 2, while the dimension of the corresponding eigenspace is just 1 (check this!), so by Theorem 7, A is not diagonalizable.

    We first calculate the eigenvalues of B. The characteristic polynomial is given by

    $$\det(B - \lambda I) = -(\lambda + 3) \cdot (\lambda - 5)^2,$$

    so the eigenvalues are $\lambda = -3$, with multiplicity 1, and $\lambda = 5$, with multiplicity 2. Now we compute the eigenspaces corresponding to $\lambda = -3$ and $\lambda = 5$.

    The eigenspace E_{-3} corresponding to $\lambda = -3$ is given by the solutions to the equation

    $$(B + 3I) \cdot x = 0,$$

    whose coefficient matrix

    $$\begin{bmatrix} -4 & -16 & 4 \\ 6 & 16 & -2 \\ 12 & 16 & 4 \end{bmatrix}$$

    is row equivalent to

    $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

    so x_3 is the only free variable, so taking $x_3 = 2$ we get that $v = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is a basis for E_{-3}.

The eigenspace E_5 corresponding to $\lambda = 5$ is given by the solutions to the equation

$$(B - 5I) \cdot x = 0,$$

whose coefficient matrix

$$\begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} 1 & 4/3 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so x_2, x_3 are free variables. Taking $x_2 = 3, x_3 = 0$ and $x_2 = 0, x_3 = 3$ we get that $w_1 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ form a basis for E_5.

Since the dimensions of the eigenspaces are equal to the multiplicities of the corresponding eigenvalues, it follows from Theorem 7 that B is diagonalizable, and we can take $P = [v\ w_1\ w_2]$ to be the matrix that diagonalizes B. More precisely, we have

$$P^{-1}BP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

12. If A is diagonalizable, then there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal. Since D is a product of invertible matrices, it is in turn invertible, and D^{-1} is clearly diagonal. It follows that

$$D^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P,$$

i.e. A^{-1} is similar to the diagonal matrix D^{-1}, so it is diagonalizable.

13. a) $T(2 - t + t^2) = 2 - t + t^2 + t^2 \cdot (2 - t + t^2) = 2 - t + 3t^2 - t^3 + t^4$.

b) We need to show that $T(p + q) = T(p) + T(q)$ and $T(c \cdot p) = c \cdot T(p)$ when $p, q \in \mathbb{P}_2$ and $c \in \mathbb{R}$. Consider then an arbitrary $c \in \mathbb{R}$, and arbitrary polynomials p, q of degree at most two. We have

$$T(p + q) = p + q + t^2(p + q) = (p + t^2p) + (q + t^2q) = T(p) + T(q)$$

and

$$T(cp) = cp + t^2 \cdot cp = c(p + t^2p) = c \cdot T(p),$$

so T is indeed linear.

c) Denote $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$, the bases for $\mathbb{P}_2$ and $\mathbb{P}_4$ respectively. We have

$$T(1) = 1 + t^2, \text{ so } [T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$T(t) = t + t^3, \text{ so } [T(t)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$T(t^2) = t^2 + t^4, \text{ so } [T(t^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

It follows that the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

d) If p is in the kernel of T, then

$$0 = T(p) = p + t^2 \cdot p = (1 + t^2)p.$$

A product of polynomials is zero if and only if one of them is zero, but since $1 + t^2 \neq 0$, it follows that p must be zero. (Equivalently, you can check that the null space of the matrix of T relative to the bases $\mathcal{B}, \mathcal{C}$ is trivial)

Since the kernel of T is 0, T is one-to-one, so a basis for its image consists of the images of the elements of the basis $\mathcal{B}$: $T(1) = 1 + t^2$, $T(t) = t + t^3$ and $T(t^2) = t^2 + t^4$.

14. If $\mathcal{B}$ is a basis of eigenvectors of A, then $[T]_{\mathcal{B}}$ is diagonal. To find such a basis, we first find the eigenvalues of A. The characteristic equation

$$\lambda^2 - 5\lambda = 0$$

has solutions $\lambda = 0$ and $\lambda = 5$, which are therefore the eigenvalues of A. A basis for the eigenspace E_0 corresponding to $\lambda = 0$ is given by the vector $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. A basis for the eigenspace E_5 corresponding to $\lambda = 5$ is given by the vector $w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. It follows that $\mathcal{B} = \{v, w\}$ is a basis in which

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}.$$

15. a) Since A is similar to B, there exist an invertible matrix P such that $P^{-1}AP = B$. Since B is a product of invertible matrices, it must be invertible. We have

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P,$$

which says that A^{-1} is similar to B^{-1}.

b) As before, there exist P such that $P^{-1}AP = B$. Squaring this equality we obtain

$$(P^{-1}AP)^2 = B^2,$$

but

$$(P^{-1}AP)^2 = P^{-1}AP \cdot P^{-1}AP = P^{-1}A^2P,$$

so that $P^{-1}A^2P = B^2$, hence A^2 is similar to B^2.

c) If B is similar to A then $P^{-1}BP = A$ for some invertible matrix P. If C is similar to A then $Q^{-1}BQ = A$ for some invertible matrix Q. It follows that

$$P^{-1}BP = Q^{-1}CQ,$$

or by multiplying with Q on the left and Q^{-1} on the right, that

$$(QP^{-1})B(PQ^{-1}) = C.$$

Note that if we let $R = PQ^{-1}$, then $R^{-1} = QP^{-1}$, so

$$R^{-1}BR = C,$$

i.e. B is similar to C.

d) We have

$$B \cdot P^{-1}x = P^{-1}AP \cdot P^{-1}x \overset{P \cdot P^{-1} = I}{=} P^{-1} \cdot Ax \overset{Ax = \lambda x}{=} P^{-1} \cdot \lambda x = \lambda \cdot P^{-1}x,$$

so $P^{-1}x$ is an eigenvector of B corresponding to the eigenvalue λ.

e) The rank of a matrix doesn't change when it is multiplied (on the left or right) with an invertible matrix. If A is similar to B, then $P^{-1}AP = B$ for some invertible matrix P. But since $B = P^{-1}AP$ is obtained from A by multiplications with invertible matrices P and P^{-1}, it follows that A and B have the same rank.