Worksheet 7

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October 8, 2010

1. Let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute the quantities

   a) $u \cdot v$, $v \cdot u$, $\dfrac{v \cdot u}{u \cdot u}$.

   b) $\left(\dfrac{u \cdot v}{v \cdot v}\right) v$.

   c) $||v||$.

   d) $||x||$.

2. Find a unit vector in the direction of $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$.

3. Find the distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

4. Determine which pairs of vectors are orthogonal

   a) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $z = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$.

   b) $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$.

5. Show that the set W of vectors orthogonal to $u = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$ is a vector space. Find a basis for W.

6. Let $W = \text{Span}\{v_1, \cdots, v_p\}$. Show that if x is orthogonal to each v_j, for $1 \leq j \leq p$, then x is orthogonal to every vector in W.

7. Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis of $\mathbb{R}^3$, where $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$. Then express $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of the u's.

8. Compute the distance from $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to the line through $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and the origin.

9. Let U be a square matrix with orthonormal columns. Explain why U is invertible.

10. Let $y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. Find the closest point to y in the subspace W spanned by v_1 and v_2. Find the distance from y to W.

11. Let $y = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, and $W = \text{Span}\{u_1\}$.

    a) Let U be the 2×1 matrix whose only column is u_1. Compute $U^T \cdot U$ and $U \cdot U^T$.

    b) Compute $\text{proj}_W(y)$ and $(U \cdot U^T) \cdot y$.

12. Let A be an $m \times n$ matrix. Prove that every vector x in $\mathbb{R}^n$ can be written in the form $x = p + u$, where p is in $\text{Row}(A)$ and u is in $\text{Nul}(A)$. Also, show that if the equation $Ax = b$ is consistent, then there is a unique p in $\text{Row}(A)$ such that $Ap = b$.

13. Find an orthogonal basis for the column space of the matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

14. Find the QR-factorization of the matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}.$$

15. Suppose that $A = QR$, where Q is an $m \times n$ matrix and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible. [*Hint*: Study the equation $Rx = 0$, and use the fact that $A = QR$]