# Worksheet 7 - Solutions

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October 8, 2010

1. a) $u \cdot v = v \cdot u = 8$. $u \cdot u = 5$, so $\dfrac{v \cdot u}{u \cdot u} = 8/5$.

   b) $v \cdot v = 52$, so $\left(\dfrac{u \cdot v}{v \cdot v}\right) v = \dfrac{8}{52} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$.

   c) $||v|| = \sqrt{v \cdot v} = 2\sqrt{13}$

   d) $||x|| = \sqrt{x \cdot x} = 7$.

2. If $v = \begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$, the unit vector in the direction of $v$ is $\frac{1}{||v||} \cdot v$. We have $||v|| = \sqrt{v \cdot v} = \sqrt{69}/4$, so

$$\frac{1}{||v||} \cdot v = \frac{1}{\sqrt{69}} \cdot \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}.$$

3. The distance between $u$ and $z$ is $||u - z|| = \sqrt{(u - z) \cdot (u - z)}$. Since $u - z = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$, we get $||u - z|| = 2\sqrt{17}$.

4. a) $u \cdot z = 0$, so $u$ and $z$ are orthogonal.

   b) $u \cdot z = 0$, so $u$ and $z$ are orthogonal.

5. We show that $W$ is closed under addition and multiplication by scalars. Let $v_1, v_2 \in W$ and $c \in \mathbb{R}$. We have $u \cdot v_1 = u \cdot v_2 = 0$, so $u \cdot (v_1 + v_2) = u \cdot v_1 + u \cdot v_2 = 0$. Also, $u \cdot (cv_1) = c(u \cdot v_1) = 0$, thus $W$ is a vector space.

   Alternatively, we note that $W$ coincides with the null space of the matrix

$$A = u^T = [5 \quad -6 \quad 7] \sim [1 \quad -6/5 \quad 7/5],$$

   so it is a vector space. $A$ has only one pivot column, and its null space consists of vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, with $x_1 = 6/5x_2 - 7/5x_3$. We get

$$x = \begin{bmatrix} 6/5x_2 - 7/5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \frac{x_2}{5} \cdot \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} - \frac{x_3}{5} \cdot \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix},$$

so a basis for $W$ is $\left\{ \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix} \right\}$.

6. Any vector $v$ in $W$ is a linear combination $v = c_1 \cdot v_1 + \cdots + c_p \cdot v_p$, for some scalars $c_1, \cdots, c_p$. We get

$$x \cdot v = x \cdot (c_1 \cdot v_1 + \cdots + c_p \cdot v_p) = c_1 \cdot (x \cdot v_1) + \cdots + c_p \cdot (x \cdot v_p) = 0,$$

since $x \cdot v_1 = \cdots = x \cdot v_p$. It follows that $x$ is orthogonal to $v$.

7. We have $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$, so $\{u_1, u_2, u_3\}$ is an orthogonal basis of $\mathbb{R}^3$. We have $u_1 \cdot u_1 = 18$, $u_2 \cdot u_2 = 9$, $u_3 \cdot u_3 = 18$, $u_1 \cdot x = 24$, $u_2 \cdot x = 3$, $u_3 \cdot x = 6$, thus

$$x = \left( \frac{u_1 \cdot x}{u_1 \cdot u_1} \right) \cdot u_1 + \left( \frac{u_2 \cdot x}{u_2 \cdot u_2} \right) \cdot u_2 + \left( \frac{u_3 \cdot x}{u_3 \cdot u_3} \right) \cdot u_3 = \frac{4}{3} \cdot u_1 + \frac{1}{3} \cdot u_2 + \frac{1}{3} \cdot u_3.$$

8. Let $u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Denote by $\widetilde{u}$ the projection of the vector $u$ on the line $\mathrm{Span}(v)$. We have

$$\widetilde{u} = \left( \frac{u \cdot v}{v \cdot v} \right) \cdot v = \frac{3}{10} \cdot v = \begin{bmatrix} 12/5 \\ 9/5 \end{bmatrix}.$$

The distance between $u$ and the line spanned by $v$ is equal to the length of the vector $u - \widetilde{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$. This is $||u - \widetilde{u}|| = 1$.

9. If $U$ has orthonormal columns, then $U^T \cdot U = I$, so $U$ is invertible with inverse $U^T$.

   Alternatively, one can notice that any set of orthonormal vectors is linearly independent. It follows that $U$ has linearly independent columns, so it is invertible.

10. The closest point to $y$ in $W$ is the point $\widetilde{y}$, the projection of $y$ on $W$. Since $\{v_1, v_2\}$ is an orthogonal basis for $W$ ($v_1 \cdot v_2 = 0$), it follows that

$$\widetilde{y} = \left( \frac{y \cdot v_1}{v_1 \cdot v_1} \right) \cdot v_1 + \left( \frac{y \cdot v_2}{v_2 \cdot v_2} \right) \cdot v_2 = 3v_1 + v_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

The distance from $y$ to $W$ is $||y - \widetilde{y}||$. We have $y - \widetilde{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$, so $||y - \widetilde{y}|| = 8$.

11. a) We have $U^T \cdot U = [u_1 \cdot u_1] = [1]$, and

$$U \cdot U^T = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix} = \frac{1}{10} \cdot \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}.$$

b) We have

$$\mathrm{proj}_W(y) = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) \cdot u_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix},$$

and

$$(U \cdot U^T) \cdot y = \frac{1}{10} \cdot \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{10} \cdot \begin{bmatrix} -20 \\ 60 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

This is not an accident: whenever $U$ is a matrix with orthonormal columns, and $W = \mathrm{Col}(U)$, we have that for any vector $y$

$$\mathrm{proj}_W(y) = (U \cdot U^T) \cdot y.$$

Can you see why?

12. It is a general fact that for any subspace $W \subset \mathbb{R}^n$ and any vector $x \in \mathbb{R}^n$, $x$ can be written as the sum of a vector $p$ in $W$ (the projection $p = \hat{x} = \mathrm{pr}_W(x)$ of $x$ on $W$), and a vector $u = x - \hat{x}$ orthogonal to $W$. We apply this in the special case $W = \mathrm{Row}(A)$.

We take $p = \mathrm{pr}_{\mathrm{Row}(A)}(x)$ and let $u = x - p$. We know that $\mathrm{Nul}(A) = \mathrm{Row}(A)^{\perp}$ (see the book for details), so, since $u$ is orthogonal to $\mathrm{Row}(A)$, $u \in \mathrm{Nul}(A)$.

Assume now that the system of equations $Ax = b$ is consistent, and denote by $x$ a particular solution. Write $x = p + u$ as above, with $p \in \mathrm{Row}(A)$ and $u \in \mathrm{Nul}(A)$. We have

$$b = Ax = A(p + u) = Ap + Au = Ap + 0 = Ap.$$

(here we have used $Au = 0$, since $u \in \mathrm{Nul}(A)$)

To see that $p$ is the unique element of $\mathrm{Row}(A)$ with the property that $Ap = b$, suppose there exist another one, and call it $p'$. We have

$$A(p - p') = Ap - Ap' = b - b = 0,$$

so $p - p' \in \mathrm{Nul}(A) = \mathrm{Row}(A)^{\perp}$. But $p - p' \in \mathrm{Row}(A)$, so

$$p - p' \in \mathrm{Row}(A) \cap \mathrm{Row}(A)^{\perp}.$$

Again it is a general fact that for any $W$, $W \cap W^{\perp} = 0$, and applying this in our case we get $p - p' = 0$, so $p = p'$. (Really, what the above relation says is that $p - p'$ is orthogonal on itself, or equivalently $||p - p'|| = \sqrt{(p - p') \cdot (p - p')} = 0$, i.e. $p - p'$ is a vector of length 0, thus $p - p' = 0$.)

13. We use the Gram-Schmidt method. We start with the vectors $v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$, which span the column space of our matrix.

We have $v_1 \cdot v_2 = -40 \neq 0$, so $v_1, v_2$ are not orthogonal. We then replace $v_2$ by $v_2 - \mathrm{pr}_{\mathrm{Span}\{v_1\}}(v_2)$, i.e. set

$$v_2' = v_2 - \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 = v_2 + 2v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}.$$

We have $v_3 \cdot v_1 = 30$ and $v_3 \cdot v_2' = -10$, so $v_3$ is not orthogonal on $\text{Span}\{v_1, v_2'\}$. We replace $v_3$ by $v_3 - \text{pr}_{\text{Span}\{v_1,v_2'\}}(v_3)$, i.e. set

$$v_3' = v_3 - \left(\frac{v_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left(\frac{v_3 \cdot v_2'}{v_2' \cdot v_2'}\right) \cdot v_2' = v_3 - \frac{3}{2}v_1 + \frac{1}{2}v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

It follows that

$$\{v_1, v_2', v_3'\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

is an orthogonal basis for the column space of our matrix.

14. We use the Gram-Schmidt method to find an orthogonal basis for the column space of our matrix (which we call $A$), and then normalize the vectors to get the orthonormal columns of the matrix $Q$. We start with the vectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}$, which span the column space of our matrix $A$.

We have $v_1 \cdot v_2 = -5 \neq 0$, so $v_1, v_2$ are not orthogonal. We then replace $v_2$ by $v_2 - \text{pr}_{\text{Span}\{v_1\}}(v_2)$, i.e. set

$$v_2' = v_2 - \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 = v_2 + v_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}.$$

We have $v_3 \cdot v_1 = 20$ and $v_3 \cdot v_2' = -12$, so $v_3$ is not orthogonal on $\text{Span}\{v_1, v_2'\}$. We replace $v_3$ by $v_3 - \text{pr}_{\text{Span}\{v_1,v_2'\}}(v_3)$, i.e. set

$$v_3' = v_3 - \left(\frac{v_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left(\frac{v_3 \cdot v_2'}{v_2' \cdot v_2'}\right) \cdot v_2' = v_3 - 4v_1 + \frac{1}{3}v_2' = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}.$$

It follows that

$$\{v_1, v_2', v_3'\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

is an orthogonal basis for the column space of our matrix. We now need to normalize in order to get an orthonormal basis. We have $||v_1|| = \sqrt{5}$, $||v_2'|| = 6$ and $||v_3'|| = 4$, so $Q$ has columns $v_1/\sqrt{5}$, $v_2'/6$ and $v_3'/4$, i.e.

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & -1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}.$$

Since $Q$ is orthogonal, the inverse of $Q$ is $Q^T$, so

$$R = Q^T \cdot A = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}.$$

15. Since the columns of $A$ are linearly independent, the equation $Ax = 0$ has only the trivial solution $x = 0$. Suppose now $R$ is not invertible. Since $R$ is a square matrix, its columns must be linearly dependent, hence the equation $Rx = 0$ must have a nontrivial solution $x_0 \neq 0$. We obtain

$$A \cdot x_0 = QR \cdot x_0 = Q \cdot (R \cdot x_0) = Q \cdot 0 = 0,$$

so $x_0$ is a nontrivial solution to the equation $Ax = 0$, contradicting the original assumption.