# Worksheet 8

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1. Find a least-squares solution of $Ax = b$, where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

2. Describe all least-squares solutions of the equation $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}.$$

3. Let $A$ be an $m \times n$ matrix. Show that $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$, and that $\mathrm{Col}(A^T) = \mathrm{Col}(A^T A)$.

4. Let $A$ be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of $A$ are linearly independent. Conversely, if the columns of $A$ are linearly independent, show that $A^T A$ is invertible.

5. Describe all the least-squares solutions of the system

$$\begin{cases} x + y = 2 \\ x + y = 4 \end{cases}$$

6. Consider the inner product on $\mathbb{P}_2$ given by evaluation at $-1, 0$ and $1$, and let $p = 3t - t^2$, $q(t) = 3 + 2t^2$. Compute $\langle p, q \rangle$, $||p||$, $||q||$, and the orthogonal projection of $r(t) = t^2$ onto the subspace spanned by $p$ and $q$.

7. If $V$ is an inner product space, and $u, v$ are vectors in $V$, show that

$$\langle u, v \rangle = \frac{1}{4}||u + v||^2 - \frac{1}{4}||u - v||^2.$$

8. Let $V = C[0, 1]$, with inner product given by

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t) dt.$$

Let $f = 5t - 3$, $g(t) = t^3 - t^2$. Compute $\langle f, g \rangle$ and $||f||$.

9. Orthogonally diagonalize the symmetric matrices

$$A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}.$$

10. Suppose that $A$ is invertible and orthogonally diagonalizable. Explain why $A^{-1}$ is also orthogonally diagonalizable.

11. Suppose $A = PRP^{-1}$, where $P$ is orthogonal and $R$ is upper triangular. Show that if $A$ is symmetric, then $R$ is symmetric and hence is actually a diagonal matrix.

12. Classify the quadratic forms

    a) $3x_1^2 - 4x_1x_2 + 6x_2^2$,

    b) $9x_1^2 - 8x_1x_2 + 3x_2^2$,

    c) $2x_1^2 + 10x_1x_2 + 2x_2^2$,

    and then make a change of variable $x = Pu$ that transforms the quadratic form into one with no cross-product terms.

13. Let $A, B$ by symmetric matrices with positive eigenvalues. Show that the eigenvalues of $A + B$ are all positive.

14. Show that if an $n \times n$ matrix $A$ is positive definite, then there is a positive definite matrix $B$ such that $A = B^T B$.