Worksheet 8 - Solutions

Claudiu Raicu

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1. We compute the normal equations

$$A^T A x = A^T b.$$

We have

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

The system of equations $(A^T A)x = A^T b$ has augmented matrix

$$\begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The unique solution is given by

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The projection of b on the column space of A is given by

$$\hat{b} = A\hat{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

2. We compute the normal equations

$$A^T A x = A^T b.$$

We have

$$A^T A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}.$$

The system of equations $(A^T A)x = A^T b$ has augmented matrix

$$\begin{bmatrix} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has x_3 as a free variable and the general solution is given by

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 + 5 \\ x_3 - 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

The projection of b on the column space of A is given by

$$\hat{b} = A\hat{x} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}.$$

3. If $x \in \mathrm{Nul}(A)$ then $Ax = 0$, hence $A^T Ax = 0$, so $x \in \mathrm{Nul}(A^T A)$. Conversely, if $x \in \mathrm{Nul}(A^T A)$, then $A^T Ax = 0$. Multiplying on the left by x^T we obtain $x^T A^T Ax = 0$, but $x^T A^T = (Ax)^T$, so we get $(Ax)^T (Ax) = 0$, i.e. $||Ax|| = \sqrt{(Ax) \cdot (Ax)} = \sqrt{0} = 0$, which says that $Ax = 0$, so $x \in \mathrm{Nul}(A)$.

   To show the equality of column spaces, note that $\mathrm{Col}(A^T) = \mathrm{Row}(A)$ and therefore

   $$\mathrm{Nul}(A) = (\mathrm{Row}(A))^{\perp} = (\mathrm{Col}(A^T))^{\perp}.$$

   A similar argument for $A^T A$, using the fact that $A^T A$ is a symmetric matrix ($(A^T A)^T = A^T A$) shows that

   $$\mathrm{Nul}(A^T A) = (\mathrm{Row}(A^T A))^{\perp} = (\mathrm{Col}(A^T A))^{\perp}.$$

   Since $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$, it follows that their orthogonal complements, i.e. $\mathrm{Col}(A^T)$ and $\mathrm{Col}(A^T A)$ have to be equal.

4. Suppose first that $A^T A$ is invertible. It follows that $\mathrm{Nul}(A^T A) = 0$, but the previous exercise implies that $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A) = 0$, i.e. the columns of A are linearly independent.

   Now, if the columns of A are linearly independent, it follows that $\mathrm{Nul}(A) = 0$. This shows that $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A) = 0$, i.e. the columns of $A^T A$ are linearly independent. But $A^T A$ is then a square matrix with linearly independent columns, so it must be invertible.

5. We write the system as $Ax = b$, with

   $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

   We use the normal equations to find the least-squares solutions of the system. We have

   $$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$$

   The system of equations $(A^T A)x = A^T b$ has augmented matrix

   $$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

   This has x_2 as a free variable, so the solutions are given by

   $$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 + 3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

   The projection of b on the column space of A is given by

   $$\hat{b} = A\hat{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

6. We have

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = -10,$$
$$||p|| = \sqrt{\langle p, p\rangle} = \sqrt{20} = 2\sqrt{5},$$
$$||q|| = \sqrt{\langle q, q\rangle} = \sqrt{59}.$$

To find an orthogonal projection of r on $W = \mathrm{Span}(p, q)$ we first compute an orthogonal basis of W. We replace q by $q_1 = q - \mathrm{pr}_{\mathrm{Span}(p)}(q)$, i.e.

$$q_1 = q - \frac{\langle q, p\rangle}{\langle p, p\rangle} \cdot p = q + \frac{1}{2}p = 3 + \frac{3}{2}t + \frac{3}{2}t^2.$$

The projection of r on W is then

$$\mathrm{proj}_W(r) = \frac{\langle r, p\rangle}{\langle p, p\rangle} \cdot p + \frac{\langle r, q_1\rangle}{\langle q_1, q_1\rangle} \cdot q_1.$$

We have $\langle r, p\rangle = -2$, $\langle r, q_1\rangle = 9$ and $\langle q_1, q_1\rangle = 9$ so

$$\mathrm{proj}_W(r) = \frac{-2}{20} \cdot p + \frac{9}{9} \cdot q_1 = 3 + \frac{6}{5}t + \frac{8}{5}t^2.$$

7. We have

$$||u + v||^2 = \langle u + v, u + v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle\,,$$
$$||u - v||^2 = \langle u - v, u - v\rangle = \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle\,.$$

Subtracting the two relations we obtain

$$||u + v||^2 - ||u - v||^2 = 2(\langle u, v\rangle + \langle v, u\rangle) = 4\,\langle u, v\rangle\,,$$

using the symmetry of the inner product ($\langle u, v\rangle = \langle v, u\rangle$). Dividing by 4 the above equality we get the desired formula.

8. We have

$$\langle f, g\rangle = \int_0^1 (5t - 3)(t^3 - t^2)dt = \int_0^1 (5t^4 - 8t^3 + 3t^2)dt = [t^5 - 2t^4 + t^3]_0^1 = 0,$$

$$\langle f, f\rangle = \int_0^1 (5t - 3)^2 dt = \left[\frac{(5t - 3)^3}{3 \cdot 5}\right]_0^1 = \frac{8 - (-27)}{15} = \frac{7}{3},$$

so $||f|| = \sqrt{7/3}$.

9. We have

$$\det(A - \lambda I) = \lambda^2 - 17\lambda$$

so the eigenvalues are 0 and 17. A basis for $E_0 = \mathrm{Nul}(A)$ consists of the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while a basis for $E_17 = \mathrm{Nul}(A - 17I)$ consists of the vector $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$. The two vectors together give an orthogonal basis of $\mathbb{R}^2$, and to get an orthonormal basis of eigenvectors of A we have to normalize them by dividing by their length. We get that

$$P = \begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ 4/\sqrt{17} & -1/\sqrt{17} \end{bmatrix}$$

is an orthogonal matrix that diagonalizez A, i.e.

$$P^T A P = D = \begin{bmatrix} 0 & 0 \\ 0 & 17 \end{bmatrix}.$$

To diagonalize B, we compute

$$\det(B - \lambda I) = (\lambda - 13)(\lambda - 7)(\lambda - 1)$$

so the eigenvalues are 1 and 7 and 13. A basis for $E_1 = \mathrm{Nul}(A)$ consists of the vector $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, a basis for $E_7 = \mathrm{Nul}(A - 7I)$ consists of the vector $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, and a basis for $E_{13} = \mathrm{Nul}(A - 13I)$ consists of the vector $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. The three eigenvectors together give an orthogonal basis of $\mathbb{R}^3$, and to get an orthonormal basis of eigenvectors of B we have to normalize them by dividing by their length. We get that

$$P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix}$$

is an orthogonal matrix that diagonalizez B, i.e.

$$P^T B P = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}.$$

10. If A is orthogonally diagonalizable, A must be a symmetric matrix. Since it is also invertible, it means that A^{-1} is also a symmetric matrix, so A^{-1} is orthogonally diagonalizable.

11. If P is orthogonal, then $P^{-1} = P^T$. Since A is symmetric $A = A^T = PR^TP^T$, so

$$PRP^T = PR^TP^T,$$

i.e. $R = R^T$ is symmeric. Since R is upper triangular and symmetric, it is also lower triangular, and hence diagonal.

12. a) $3x_1^2 - 4x_1x_2 + 6x_2^2 = x^T Ax$, with

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}.$$

The eigenvalues of A are solutions to the characteristic equation $\lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7) = 0$, i.e. they are 2 and 7, which are positive numbers, hence the quadratic form is positive definite. E_2 has a basis consisting of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and E_7 has a basis consisting of the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Normalizing, we get

$$P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

b) $9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T Ax$, with

$$A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}.$$

The eigenvalues of A are solutions to the characteristic equation $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1) = 0$, i.e. they are 11 and 1, which are positive numbers, hence the quadratic form is positive definite. E_{11} has a basis consisting of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and E_1 has a basis consisting of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Normalizing, we get

$$P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

c) $2x_1^2 + 10x_1x_2 + 2x_2^2 = x^T Ax$, with

$$A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}.$$

The eigenvalues of A are solutions to the characteristic equation $\lambda^2 - 4\lambda - 21 = (\lambda - 7)(\lambda + 3) = 0$, i.e. they are 7 and -3, which have opposite signs, hence the quadratic form is indefinite. E_7 has a basis consisting of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and E_{-3} has a basis consisting of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Normalizing, we get

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

13. If A, B have positive eigenvalues, the quadratic forms $x^T Ax$ and $x^T Bx$ are positive definite, i.e. $x^T Ax > 0$ and $x^T Bx > 0$ for all vectors $x \neq 0$. It follows that the same holds for the sum of the two quadratic forms

$$x^T Ax + x^T Bx = x^T (A + B)x,$$

i.e. $x^T (A + B)x$ is positive definite, so $A + B$ has only positive eigenvalues.

14. If A is positive definite, then it is orthogonally diagonalizable to a diagonal matrix with positive entries. Write

$$P^T AP = D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

Since D has positive entries, we can take their square roots to get a diagonal matrix E, with

$$E = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}.$$

We have $E^T = E$, so $E^T E = E^2 = D$, which yields

$$A = PDP^T = PEE^T P^T = (PE)(PE)^T.$$

If we let $B = (PE)^T$, we get $B^T B = A$.