Math 54, Fall '10
Quiz 10, November 10

1. (3 points) Express the system of higher-order differential equations

$$\begin{cases} x'' + 3x + 2y = 0 \\ y'' - 2x = 0 \end{cases}$$

as a matrix system in normal form.

Solution. We let $x_1 = x$, $x_2 = x'$, $x_3 = y$ and $x_4 = y'$. The system of equations becomes

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 - 2x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 \end{cases}$$

$\square$

2. (3 points) Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix}.$$

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8,$$

which factors as

$$-(\lambda - 4)(\lambda - 2)(\lambda - 1),$$

so the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$. The corresponding eigenvectors are

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}.$$

A fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ has columns $e^{\lambda_1 t} \cdot u_1, e^{\lambda_2 t} \cdot u_2, e^{\lambda_3 t} \cdot u_3$:

$$\begin{bmatrix} e^t & e^{2t} & e^{4t} \\ e^t & 2e^{2t} & 4e^{4t} \\ e^t & 4e^{2t} & 16e^{4t} \end{bmatrix}.$$

Note that the system $\mathbf{x}' = A\mathbf{x}$ is the representation of the 3rd order linear homogeneous differential equation $y''' - 7y'' + 14y' - 8y = 0$ as a linear system in normal form. It follows that the entries in the first row of the fundamental matrix above give a basis for the solution set of this equation. $\square$