Math 54, Fall '10
Quiz 11, November 17

1. (3 points) Use undetermined coefficients to find a particular solution of the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. Since 0 is an eigenvalue of A, we cannot hope to find a particular solution $\mathbf{x}_p = \mathbf{a}$, i.e. a constant vector. We look for a particular solution of the form $\mathbf{x}_p = \mathbf{a} + t\mathbf{b}$. We get $\mathbf{x}_p' = \mathbf{b}$, so the equation $\mathbf{x}_p' = A\mathbf{x}_p + \mathbf{f}$ becomes $\mathbf{b} = A(\mathbf{a} + t\mathbf{b}) + \mathbf{f}$, i.e. $\mathbf{b} = A\mathbf{a} + \mathbf{f}$ and $A\mathbf{b} = 0$. Substituting $\mathbf{b}$ in the second equation yields

$$A \cdot (A\mathbf{a} + \mathbf{f}) = 0.$$

If we write $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, we get, using the fact that $A^2 = A$, that $A\mathbf{a} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, i.e. $a_2 = -1$ and a_1 is arbitrary. We take $a_1 = 0$ and compute $\mathbf{b}$ using the formula $\mathbf{b} = A\mathbf{a} + \mathbf{f}$. This yields $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that a particular solution for $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ is given by

$$\mathbf{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

□

2. (3 points) Use variation of parameters to find a general solution of the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}, \quad f = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}.$$

Bonus (1 point): What is e^{At}?

Solution. The eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ are $\lambda_1 = -1$, $\lambda_2 = 1$, with corresponding eigenvectors $u_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A fundamental matrix is therefore

$$X(t) = \begin{bmatrix} -e^{-t} & -e^t \\ 3e^{-t} & e^t \end{bmatrix}, \text{ with } X^{-1}(t) = \frac{1}{2}\begin{bmatrix} e^t & e^t \\ -3e^{-t} & -e^{-t} \end{bmatrix}.$$

We get

$$X^{-1}(t) \cdot f(t) = \begin{bmatrix} 3e^{2t} \\ -5 \end{bmatrix},$$

thus

$$\int X^{-1}(t) \cdot f(t)dt = \begin{bmatrix} 3e^{2t}/2 \\ -5t \end{bmatrix},$$

and

$$X(t)\int X^{-1}(t)\cdot f(t)dt = \begin{bmatrix} -3e^t/2+5te^t \\ 9e^t/2-5te^t \end{bmatrix}.$$

It follows that the general solution is given by

$$c_1\begin{bmatrix} -e^{-t} \\ 3e^{-t} \end{bmatrix} + c_2\begin{bmatrix} -e^t \\ e^t \end{bmatrix} + \begin{bmatrix} -3e^t/2+5te^t \\ 9e^t/2-5te^t \end{bmatrix}.$$

Bonus:

$$e^{At} = X(t)\cdot X^{-1}(0) = \begin{bmatrix} -e^{-t} & -e^t \\ 3e^{-t} & e^t \end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix} = \frac{1}{2}\cdot\begin{bmatrix} 3e^t-e^{-t} & e^t-e^{-t} \\ 3(e^{-t}-e^t) & 3e^{-t}-e^t \end{bmatrix}.$$

$\square$