Math 54, Fall '10
Quiz 12, December 3

1. (3 points) a) Show that the function $f(x) = x^2$ has the Fourier series, on $-\pi < x < \pi$,

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

b) Use part (a) to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

*Solution.* a) Since $f(x) = x^2$ is an even function, the sine coefficients $b_n$ must be zero. The cosine coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx.$$

We get

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3},$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \cos(nx) dx = \frac{4 \cdot (-1)^n}{\pi n^2}.$$

It follows that the Fourier series of $f$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

b) Since $f$ and $f'$ are continuous, we get by plugging in $x = 0$ that

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n \cdot 0) = f(0) = 0,$$

i.e.

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0,$$

which after dividing by 4 and rearranging terms is equivalent to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

□

2. Solve the vibrating string problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = 9 \cdot \dfrac{\partial^2 u}{\partial x^2}(x,t), \ 0 < x < \pi, \ t > 0 \\ u(0,t) = u(\pi,t) = 0, \ t \geq 0 \\ u(x,0) = 6\sin(2x) + 2\sin(6x), \ 0 \leq x \leq \pi, \\ \dfrac{\partial u}{\partial t}(x,0) = 27\sin(9x) - 45\sin(15x), \ 0 \leq x \leq \pi. \end{cases}$$

*Solution.* We look for a solution

$$u(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$

where $L = \pi$ and $\alpha = 3$, i.e.

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(3nt) + b_n \sin(3nt)) \sin(nx).$$

The initial condition $u(x,0) = 6\sin(2x) + 2\sin(6x)$ implies

$$\sum_{n=1}^{\infty} a_n \sin(nx) = 6\sin(2x) + 2\sin(6x),$$

i.e. $a_2 = 6$, $a_6 = 2$ and $a_n = 0$ for $n \neq 2, 6$.

The initial condition $\dfrac{\partial u}{\partial t}(x,0) = 27\sin(9x) - 45\sin(15x)$ implies

$$\sum_{n=1}^{\infty} 3nb_n \sin(nx) = 27\sin(9x) - 45\sin(15x),$$

i.e. $3 \cdot 9 \cdot b_9 = 27$, $3 \cdot 15 \cdot b_{15} = -45$, and $3nb_n = 0$ for $n \neq 9, 15$. This yields $b_9 = 1$, $b_{15} = -1$, and $b_n = 0$ for $n \neq 9, 15$.

It follows that

$$u(x,t) = 6\cos(6t)\sin(2x) + 2\cos(18t)\sin(6x) + \sin(27t)\sin(9x) - \sin(45t)\sin(15x).$$

□