Math 54, Fall '10
Quiz 2, September 8

1. (2 points) If v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^4$ and $\{v_1, v_2, v_3\}$ is linearly dependent, show that $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent.

   First solution. Since v_1, v_2, v_3 are linearly dependent vectors, we can find scalars c_1, c_2, c_3, not all zero, such that
   $$c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 = 0.$$
   Taking $c_4 = 0$, it is still true that not all c_1, c_2, c_3, c_4 are equal to zero, and we have
   $$c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 + c_4 \cdot v_4 = c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 = 0,$$
   so v_1, v_2, v_3, v_4 are linearly dependent vectors. $\square$

   Second solution. One can think of the vectors v_1, v_2, v_3, v_4 as the columns of a 4×4 matrix A. The condition that the set $\{v_1, v_2, v_3\}$ is linearly dependent can be translated into saying that at least one of the first three columns of A is not a pivot column. Since not all the columns of A are pivot columns, the system of equations $Ax = 0$ has free variables. It is obviously consistent ($x = 0$ is a solution), so it has infinitely many solutions. This implies that the columns of A are linearly dependent. $\square$

2. (4 points) Let
   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$
   Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right and on the left. Find a 3×3 diagonal matrix B (with $B \neq I$) such that $AB = BA$.

   Solution. We have
   $$A \cdot D = \begin{bmatrix} 1 \cdot 2 & 1 \cdot 3 & 1 \cdot 4 \\ 1 \cdot 2 & 2 \cdot 3 & 3 \cdot 4 \\ 1 \cdot 2 & 4 \cdot 3 & 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 2 & 12 & 20 \end{bmatrix},$$
   so right-multiplication by the diagonal matrix D multiplies the columns of A by the corresponding diagonal entries of D. Also,
   $$D \cdot A = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 1 & 2 \cdot 1 \\ 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 4 \cdot 1 & 4 \cdot 4 & 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 4 & 16 & 20 \end{bmatrix},$$
   so left-multiplication by the diagonal matrix D multiplies the rows of A by the corresponding diagonal entries of D.

   If we choose B to be any multiple of the identity matrix I_3, i.e. a diagonal matrix with all entries equal, the above description of left- and right- multiplication by diagonal matrices shows that left-multiplication by B has the same effect as right-multiplication by B. Below are two examples of matrices B that do the job:

$$B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = -3 \cdot I_3, \text{ or } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \cdot I_3.$$

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