Math 54, Fall '10
Quiz 3, September 15

1. (3 points) Find the determinant and the first and last columns of the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1^1 & 2^1 & 3^1 & 4^1 \\ 1^2 & 2^2 & 3^2 & 4^2 \\ 1^3 & 2^3 & 3^3 & 4^3 \end{bmatrix}.$$

*Solution.* We augment the matrix by the first and last columns of the identity matrix and row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 \\ 1 & 4 & 9 & 16 & 0 & 0 \\ 1 & 8 & 27 & 64 & 0 & 1 \end{bmatrix} \underset{\substack{R_2=R_2-R_1 \\ R_3=R_3-R_1 \\ R_4=R_4-R_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 8 & 15 & -1 & 0 \\ 0 & 7 & 26 & 63 & -1 & 1 \end{bmatrix} \underset{\substack{R_3=R_3-3R_2 \\ R_4=R_4-4R_2}}{\sim}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 12 & 42 & 6 & 1 \end{bmatrix} \underset{R_4=R_4-6R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 0 & 6 & -6 & 1 \end{bmatrix}.$$

At this point, we can compute the determinant as the product of the pivots:

$$\det(A) = 1 \cdot 1 \cdot 2 \cdot 6 = 12.$$

We continue the row reduction algorithm:

$$\underset{\substack{R_3=R_3/2 \\ R_4=R_4/6}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1/6 \end{bmatrix} \underset{\substack{R_3=R_3-3R_4 \\ R_2=R_2-3R_4 \\ R_1=R_1-R_4}}{\sim}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 2 & -1/6 \\ 0 & 1 & 2 & 0 & 2 & -1/2 \\ 0 & 0 & 1 & 0 & 4 & -1/2 \\ 0 & 0 & 0 & 1 & -1 & 1/6 \end{bmatrix} \underset{\substack{R_2=R_2-2R_3 \\ R_1=R_1-R_3}}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 & -2 & 1/3 \\ 0 & 1 & 0 & 0 & -6 & 1/2 \\ 0 & 0 & 1 & 0 & 4 & -1/2 \\ 0 & 0 & 0 & 1 & -1 & 1/6 \end{bmatrix} \underset{R_1=R_1-R_2}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 & -1/6 \\ 0 & 1 & 0 & 0 & -6 & 1/2 \\ 0 & 0 & 1 & 0 & 4 & -1/2 \\ 0 & 0 & 0 & 1 & -1 & 1/6 \end{bmatrix}.$$

It follows that the first and last columns of $A^{-1}$ are

$$\begin{bmatrix} 4 \\ -6 \\ 4 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1/6 \\ 1/2 \\ -1/2 \\ 1/6 \end{bmatrix}.$$

*Remark.* Notice that the negatives of the ratios between the entries in the first and last columns of $A^{-1}$ are $24, 12, 8$ and $6$, i.e. $2 \cdot 3 \cdot 4$, $1 \cdot 3 \cdot 4$, $1 \cdot 2 \cdot 4$ and $1 \cdot 2 \cdot 3$. If you replace $1, 2, 3, 4$ with arbitrary distinct numbers $a, b, c, d$, this will remain true. That is the ratios will be $-b \cdot c \cdot d$, $-a \cdot c \cdot d$, $-a \cdot b \cdot d$ and $-a \cdot b \cdot c$. Can you see why? Try it in one example! $\square$

*Alternative solution.* We consider the general case when $1, 2, 3, 4$ are replaced by distinct numbers $a, b, c, d$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}.$$

Let's first compute the determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} \overset{\substack{R_4 = R_4 - aR_3 \\ R_3 = R_3 - aR_2 \\ R_2 = R_2 - aR_1}}{=} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2 - b^1 \cdot a & c^2 - c^1 \cdot a & d^2 - d^1 \cdot a \\ 0 & b^3 - b^2 \cdot a & c^3 - c^2 \cdot a & d^3 - d^2 \cdot a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & (b-a) \cdot b^1 & (c-a) \cdot c^1 & (d-a) \cdot d^1 \\ 0 & (b-a) \cdot b^2 & (c-a) \cdot c^2 & (d-a) \cdot d^2 \end{vmatrix} = \begin{vmatrix} b-a & c-a & d-a \\ (b-a) \cdot b^1 & (c-a) \cdot c^1 & (d-a) \cdot d^1 \\ (b-a) \cdot b^2 & (c-a) \cdot c^2 & (d-a) \cdot d^2 \end{vmatrix},$$

where the last equality follows by cofactor expansion along the first column. Now we can take out the factors $(b-a), (c-a), (d-a)$ from the columns of the last matrix, and we get

$$\begin{vmatrix} b-a & c-a & d-a \\ (b-a) \cdot b^1 & (c-a) \cdot c^1 & (d-a) \cdot d^1 \\ (b-a) \cdot b^2 & (c-a) \cdot c^2 & (d-a) \cdot d^2 \end{vmatrix} = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b^1 & c^1 & d^1 \\ b^2 & c^2 & d^2 \end{vmatrix}$$

We've seen in class (and you can obtain this using the same trick as above) that

$$\begin{vmatrix} 1 & 1 & 1 \\ b^1 & c^1 & d^1 \\ b^2 & c^2 & d^2 \end{vmatrix} = (c-b)(d-b)(d-c).$$

It follows that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c),$$

i.e. the product of all the differences between the numbers $a, b, c, d$ (notice the similarity with the $3 \times 3$ Vandermonde determinant discussed in class). If we set $a = 1, b = 2, c = 3, d = 4$ we obtain the determinant equal to $1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 12$, which agrees with the calculation of the previous solution.

We have seen that the entry of $A^{-1}$ in row $i$ and column $j$ is given by the formula $(-1)^{i+j} C_{j,i}/\det(A)$, where $C_{j,i}$ is the determinant of the submatrix $A_{j,i}$ obtained from $A$ by leaving out the $j$-th row and $i$-th column.

In particular, if we're interested in the first column of $A^{-1}$, we want to compute the numbers $C_{1,i}$, i.e. the determinants of the submatrices of $A$ obtained by leaving out its first row and some column. Notice that all these submatrices look very much the same. Let's calculate $C_{1,1}$:

$$C_{1,1} = \det(A_{1,1}) = \begin{vmatrix} b^1 & c^1 & d^1 \\ b^2 & c^2 & d^2 \\ b^3 & c^3 & d^3 \end{vmatrix}.$$

Taking out the factors $b, c, d$ from the columns of $A_{1,1}$ we get

$$C_{1,1} = bcd \begin{vmatrix} 1 & 1 & 1 \\ b^1 & c^1 & d^1 \\ b^2 & c^2 & d^2 \end{vmatrix} = bcd(c-b)(d-b)(d-c).$$

We get that the entry in the first row and column of $A^{-1}$ is equal to

$$\frac{(-1)^{1+1}bcd(c-b)(d-b)(d-c)}{(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)} = \frac{bcd}{(b-a)(c-a)(d-a)}.$$

In a similar way we obtain the other entries in the first column of $A^{-1}$:

$$\begin{bmatrix} \frac{bcd}{(b-a)(c-a)(d-a)} \\ \frac{acd}{(a-b)(c-b)(d-b)} \\ \frac{abd}{(a-c)(b-c)(d-c)} \\ \frac{abc}{(a-d)(b-d)(c-d)} \end{bmatrix}.$$

(notice the symmetry between $a, b, c, d$). Let's check that this agrees with the calculations in the first solution: if we set $a = 1, b = 2, c = 3, d = 4$ one easily sees that the column vector above becomes

$$\begin{bmatrix} 4 \\ -6 \\ 4 \\ -1 \end{bmatrix}.$$

Now if we want the last column of $A^{-1}$, we have to compute the numbers $C_{4,i}$, i.e. the determinants of the submatrices of $A$ obtained by leaving out its last row and some column. Notice that all these submatrices are Vandermonde matrices. Let's calculate $C_{4,2}$:

$$C_{4,2} = \det(A_{4,2}) = \begin{vmatrix} 1 & 1 & 1 \\ a^1 & c^1 & d^1 \\ a^2 & c^2 & d^2 \end{vmatrix} = (c-a)(d-a)(d-c).$$

We get that the entry in the second row and fourth column of $A^{-1}$ is equal to

$$\frac{(-1)^{4+2}(c-a)(d-a)(d-c)}{(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)} = \frac{1}{(b-a)(c-b)(d-b)} = \frac{-1}{(a-b)(c-b)(d-b)}.$$

In a similar way we obtain the other entries in the last column of $A^{-1}$:

$$\begin{bmatrix} \frac{-1}{(b-a)(c-a)(d-a)} \\ \frac{-1}{(a-b)(c-b)(d-b)} \\ \frac{-1}{(a-c)(b-c)(d-c)} \\ \frac{-1}{(a-d)(b-d)(c-d)} \end{bmatrix}.$$

(notice again the symmetry between $a, b, c, d$). Let's check that this agrees with the calculations in the first solution: if we set $a = 1, b = 2, c = 3, d = 4$ one easily sees that the column vector above becomes

$$\begin{bmatrix} -1/6 \\ 1/2 \\ -1/2 \\ 1/6 \end{bmatrix}.$$

From the description of the first and last columns of $A^{-1}$ you can now easily check that the remark at the end of the first solution holds. $\square$

2. (3 points) Find the area of the parallelogram with vertices

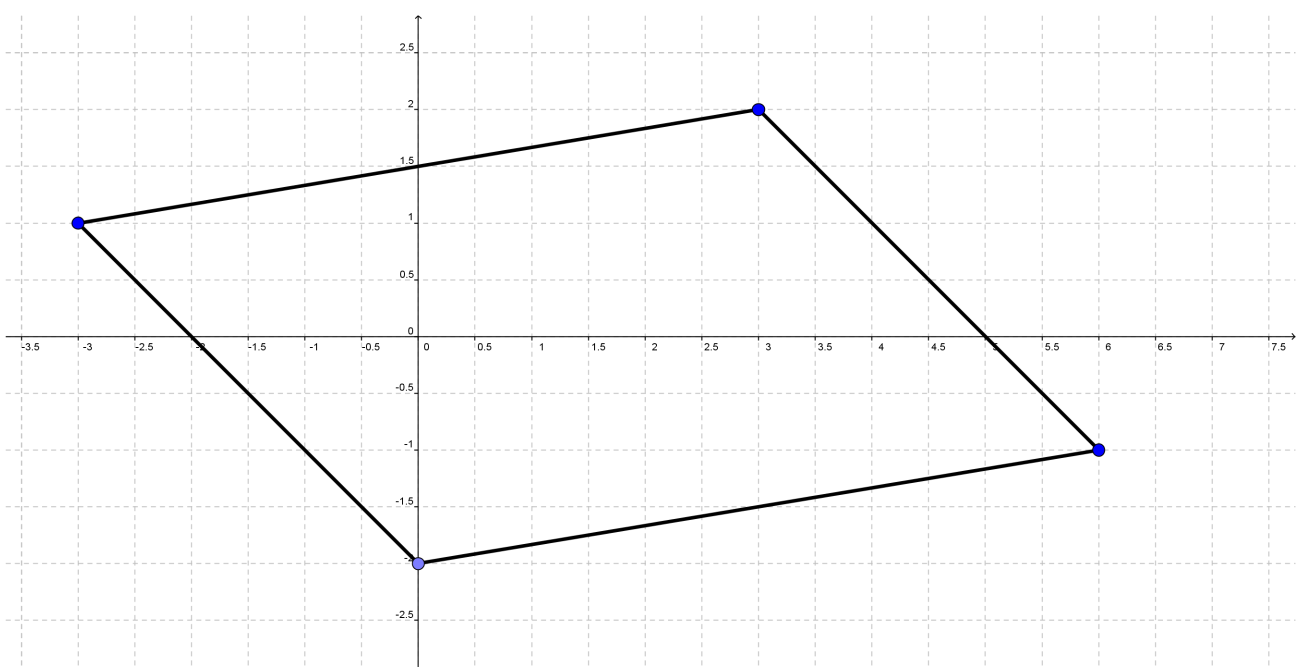
$$(0, -2), \quad (6, -1), \quad (-3, 1), \quad (3, 2).$$

*Solution.* The sides of the paralelogram have direction vectors

$$(6, -1) - (0, -2) = (6, 1)$$

and

$$(-3, 1) - (0, -2) = (-3, 3).$$



The area of the paralelogram is given by the absolute value of the determinant of the matrix

$$\begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}.$$

This is

$$|6 \cdot 3 - 1 \cdot (-3)| = 21.$$

$\square$