Math 54, Fall '10
Quiz 4, September 22

1. (3 points) Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

Find a basis for the span of $\{v_1, v_2, v_3, v_4, v_5\}$.

Solution. We can think of v_1, v_2, v_3, v_4, v_5 as the columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 6 & 0 \\ 0 & 1 & -3 & -1 & 3 \\ 0 & -1 & 3 & 2 & -1 \\ 1 & 1 & -4 & -1 & 1 \end{bmatrix}.$$

Then the span of $\{v_1, v_2, v_3, v_4, v_5\}$ is just the column space of A, $\mathrm{col}(A)$. A basis for this space is the set of pivot columns, and to find these we need to compute the echelon form of A. We have

$$A \overset{R_4=R_4-R_1}{\sim} \begin{bmatrix} 1 & -2 & 5 & 6 & 0 \\ 0 & 1 & -3 & -1 & 3 \\ 0 & -1 & 3 & 2 & -1 \\ 0 & 3 & -9 & -7 & 1 \end{bmatrix} \overset{\substack{R_3=R_3+R_2 \\ R_4=R_4-3R_2}}{\sim} \begin{bmatrix} 1 & -2 & 5 & 6 & 0 \\ 0 & 1 & -3 & -1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -4 & -8 \end{bmatrix}$$

$$\overset{R_4=R_4+4R_3}{\sim} \begin{bmatrix} 1 & -2 & 5 & 6 & 0 \\ 0 & 1 & -3 & -1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the 1st, 2nd and 4th columns of A form a basis for $\mathrm{col}(A)$, so $B = \{v_1, v_2, v_4\}$ is a basis for $\mathrm{Span}\{v_1, v_2, v_3, v_4, v_5\}$. □

2. (3 points) Let $M_{2\times2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$.

a) Show that T is a linear transformation.

b) Let B be any element of $M_{2\times2}$ such that $B^T = B$. Find an A in $M_{2\times2}$ such that $T(A) = B$.

c) Describe the kernel of T.

Solution. a) To check that T is linear it suffices to check that T preserves sums and multiplication by scalars (which in particular implies that $T(0) = 0$).

Sums:

$$T(A+B) = (A+B)+(A+B)^T = A+B+A^T+B^T = (A+A^T)+(B+B^T) = T(A)+T(B).$$

Multiplication by scalars:

$$T(c \cdot A) = (c \cdot A)^T = c \cdot A^T = c \cdot T(A).$$

b) Take $A = \frac{1}{2}B$. Then $A^T = \frac{1}{2}B^T = \frac{1}{2}B$, so

$$T(A) = A + A^T = \frac{1}{2}B + \frac{1}{2}B = B.$$

c) A matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is in the kernel of T if and only if

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix},$$

if and only if $2a = b + c = 2d = 0$, i.e. $a = d = 0$ and $c = -b$. So the kernel of T is the set

$$\ker(T) = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\}.$$

A basis for $\ker(T)$ consists of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$\square$