Math 54, Fall '10
Quiz 5, September 29

1. (3 points) Find the dimensions of Nul(A) and Col(A), and the rank of A^T, where A is the matrix

$$\begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 1 & 3 & -4 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. We first put A into echelon form:

$$A \overset{R_3=R_3-R_1}{\sim} \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & -1 & 5 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A has 3 pivot columns, so

$$\dim(\mathrm{Col}(A)) = 3,$$

which is also equal to the rank of A. The rank of the transpose of any matrix is equal to the rank of the matrix, so

$$\mathrm{rank}(A^T) = 3.$$

The dimension of the null space of a matrix equals the number of non-pivot columns, in our case $6 - 3$, so

$$\dim(\mathrm{Nul}(A)) = 3.$$

□

2. (3 points) In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Solution. Denote the basis elements of $\mathcal{B}$ by b_1, b_2, b_3, and the basis elements of $\mathcal{C}$ by c_1, c_2, c_3. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of b_1, b_2, b_3, namely

$$[b_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad [b_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \quad [b_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

It follows that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}.$$

We would like to calculate the $\mathcal{B}$-coordinate vector $[p]_{\mathcal{B}}$ of $p = t^2$. We have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [p]_{\mathcal{B}} = [p]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so in order to find $[p]_{\mathcal{B}}$ we need to solve the system of equations whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix}.$$

Row-reducing, we obtain that

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

so

$$[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

This means that $p = 3b_1 - 2b_2 + b_3$, or

$$t^2 = 3 \cdot (1 - 3t^2) - 2 \cdot (2 + t - 5t^2) + 1 \cdot (1 + 2t).$$

$\square$