Math 54, Fall '10
Quiz 6, October 6

1. (3 points) Compute $A^8$, where

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$

*First solution.* We first (try to) diagonalize $A$: its eigenvalues are the roots of the characteristic equation

$$0 = \det(A - \lambda \cdot I) = \lambda^2 - (4-1)\lambda + (4 \cdot (-1) - 2 \cdot (-3)) = \lambda^2 - 3\lambda + 2 = (\lambda - 1) \cdot (\lambda - 2).$$

We get $\lambda_1 = 1$ and $\lambda_2 = 2$. Since the eigenvalues of $A$ are distinct, $A$ is diagonalizable, and the matrix $P$ that realizes the diagonalization has columns $v_1, v_2$, where $v_1$ is an eigenvector corresponding to the eigenvalue 1, and $v_2$ is an eigenvector corresponding to the eigenvalue 2.

$v_1$ is a generator of the null space of $A - I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$, so we can take $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$v_2$ is a generator of the null space of $A - 2I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$, so we can take $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

It follows that for

$$P = [v_1\ v_2] = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix},\ P^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix},\ \text{and } P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We get $A = PDP^{-1}$ and

$$\begin{aligned} A^8 = PD^8P^{-1} &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1^8 & 0 \\ 0 & 2^8 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \cdot 2^8 \\ 1 & 2^9 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 + 3 \cdot 2^8 & 3 - 3 \cdot 2^8 \\ -2 + 2^9 & 3 - 2^9 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}. \end{aligned}$$

□

*Second solution.* We have

$$A^2 = \begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix},$$

$$A^4 = (A^2)^2 = \begin{bmatrix} 46 & -45 \\ 30 & -29 \end{bmatrix},$$

$$A^8 = (A^4)^2 = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}.$$

□

2. (3 points) Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ defined by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

a) Find the image under $T$ of $\mathbf{p}(t) = 5 + 3t$.

b) Find the matrix for $T$ relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

c) What is the kernel of $T$?

*Solution.* a) $T(5+3t) = \begin{bmatrix} 5+3\cdot(-1) \\ 5+3\cdot 0 \\ 5+3\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

b) We need to evalate the images via $T$ of the elments of the basis $\mathcal{B} = \{1, t, t^2\}$, and write them in the coordinate system determined by the standard basis $\mathcal{C}$ of $\mathbb{R}^3$. Since any given vector in $\mathbb{R}^3$ is already represented in the standard basis, the last part is trivial. We have

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ so } [T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ so } [T(t)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ so } [T(t^2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

It follows that the matrix of $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$A = \big[[T(1)]_{\mathcal{C}}\ [T(t)]_{\mathcal{C}}\ [T(t^2)]_{\mathcal{C}}\big] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

c) If $\mathbf{p}$ is a polynomial in the kernel of $T$, then $\mathbf{p}(-1) = \mathbf{p}(0) = \mathbf{p}(1) = 0$. This means that $\mathbf{p}$ is a polynomial of degree at most two, having three roots: $-1, 0$ and $1$. This is only possible if $\mathbf{p}$ is the zero polynomial, so $T$ is one-to-one and the kernel of $T$ is trivial.

Alternatively, we compute the echelon form of the matrix $A$:

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

It follows that $A$ has no nonpivot columns, i.e. its null space is 0. $\square$