Math 54, Fall '10
Quiz 8, October 27

1. (3 points) Determine if the functions $y_1(t) = e^{2t}$ and $y_2(t) = e^t$ are linearly dependent on the interval $(-\infty, \infty)$.

   Solution. If y_1, y_2 were dependent, then there would exist scalars c_1, c_2, not both equal to zero, such that $c_1y_1 + c_2y_2 = 0$. If $c_1 = 0$, then $c_2 \neq 0$ and hence $y_2 = 0$, which is not the case. Therefore we may assume that $c_1 \neq 0$ and divide by c_1 the equality $c_1y_1 + c_2y_2 = 0$. If we let $c = -c_2/c_1$ then we obtain $y_1 = cy_2$, i.e. $e^{2t} = ce^t$. Dividing by e^t yields $e^t = c$, which is not possible, since e^t is not a constant function. It follows that y_1, y_2 are linearly independent. □

   Alternative solution. y_1, y_2 are solutions of the second order homogeneous differential equation $y'' - 3y' + 2y = 0$, so we can use the Wronskian to test the dependece of y_1, y_2. We have
$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^t \\ 2e^{2t} & e^t \end{vmatrix} = -e^{3t}.$$
   Since $e^{3t} > 0$ for all t, $W(y_1, y_2)$ is never zero, hence y_1, y_2 are linearly independent. □

2. (3 points) Find three linearly independent solutions to the homogeneous differential equation
$$y^{iv} + 13y'' + 36y = 0.$$

   Solution. The auxiliary equation $r^4 + 13r^2 + 36 = 0$ is equivalent to $(r^2 + 4)(r^2 + 9) = 0$, so it has solutions $r_1 = 2i$, $r_2 = -2i$, $r_3 = 3i$ and $r_4 = -3i$. It follows that the general solution of the equation $y^{iv} + 13y'' + 36y = 0$ is given by
$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos(3t) + c_4 \sin(3t).$$
   It follows that $\{\cos(2t), \sin(2t), \cos(3t), \sin(3t)\}$ is a basis for the space of solutions of the equation $y^{iv} + 13y'' + 36y = 0$, so any three of them would form a linearly independent set. □