Math 54, Fall '10
Quiz 9, November 3

1. (3 points) Solve the differential equation using the method of undetermined coefficients.
$$y'' + y = e^x + x^3.$$

Solution. We first solve the complementary equation $y'' + y = 0$. This has auxiliary equation $r^2 + 1$ with roots $\pm i$, so its solutions are given by
$$y_c = c_1 \cos(x) + c_2 \sin(x).$$
Using the superposition principle, we split the problem of finding a particular solution to our original equation into two parts, corresponding to the two terms e^x and x^3. We look for y_{p_1}, y_{p_2} solutions of
$$y'' + y = e^x \text{ and } y'' + y = x^3$$
respectively. Taking
$$y_{p_1} = Ae^x$$
we get
$$Ae^x + Ae^x = e^x,$$
so $A = 1/2$ and $y_{p_1} = e^x/2$. Taking
$$y_{p_2} = Bx^3 + Cx^2 + Dx + E$$
we get
$$(6Bx + 2C) + (Bx^3 + Cx^2 + Dx + E) = x^3.$$
Equating the coefficients of the various powers of x we obtain
$$2C + E = 0, \quad 6B + D = 0, \quad C = 0, \quad B = 1$$
which gives $D = -6$ and $E = 0$, so
$$y_{p_2} = x^3 - 6x.$$
It follows that a particular solution to our equation is given by
$$y_p = y_{p_1} + y_{p_2} = \frac{e^x}{2} + x^3 - 6x$$
and the general solution is
$$y = y_p + y_c = \frac{e^x}{2} + x^3 - 6x + c_1 \cos(x) + c_2 \sin(x).$$
□

2. (3 points) Find a particular solution of the variable coefficient equation
$$t^2 y'' - 4ty' + 6y = t^3 + 1,$$
given that $y_1 = t^2$ and $y_2 = t^3$ are linearly independent solutions of the corresponding homogeneous equation.

Solution. We use the method of variation of parameters. We search for y_p of the form $v_1y_1 + v_2y_2$, with v_1, v_2 functions whose derivatives satisfy the system of equations

$$\begin{cases} y_1v_1' + y_2v_2' = 0 \\ y_1'v_1' + y_2'v_2' = \dfrac{g(t)}{a(t)} \end{cases}$$

Substituting y_1, y_2, $g(t) = t^3 + 1$ and $a(t) = t^2$ in the above system we obtain

$$\begin{cases} t^2v_1' + t^3v_2' = 0 \\ 2tv_1' + 3t^2v_2' = \dfrac{t^3+1}{t^2} \end{cases}$$

The first equation yields $v_1' = -tv_2'$, which combined with the second equation gives

$$t^2v_2' = \frac{t^3+1}{t^2}.$$

It follows that

$$v_2' = \frac{t^3+1}{t^4} = t^{-1} + t^{-4}, \quad v_1' = -tv_2' = -1 - t^{-3},$$

i.e.

$$v_2 = \ln|t| - \frac{1}{3}t^{-3}, \quad v_1 = -t + \frac{t^{-2}}{2}.$$

This shows that

$$y_p = t^2\left(-t + \frac{t^{-2}}{2}\right) + t^3\left(\ln|t| - \frac{1}{3}t^{-3}\right) = -t^3 + \frac{1}{2} + t^3\ln|t| - \frac{1}{3} = t^3\ln|t| - t^3 + \frac{1}{6}.$$

□