SCHURRINGS: A PACKAGE FOR COMPUTING WITH SYMMETRIC FUNCTIONS

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Abstract. We describe a software package facilitating computations with symmetric functions, with an emphasis on the representation theory of general linear and symmetric groups. As an application, we implement a heuristic method for approximating equivariant resolutions of modules over polynomial rings with an action of a product of a combination of general linear and symmetric groups.

1. Introduction

The theory of symmetric functions is a branch of algebraic combinatorics which makes an appearance in numerous areas of mathematics: representation theory, algebraic geometry, Galois theory, statistics etc. The need for software that implements fast computations with symmetric functions is thus a natural consequence of their ubiquity. Stembridge's SF Package [Ste95] offers great capabilities for dealing with symmetric functions, with an emphasis on the algebraic combinatorics point of view. We propose a new package, the SchurRings package in Macaulay2 [GS], highlighting a representation theoretic perspective on symmetric functions.

Specifically, our approach to the theory of symmetric polynomials is focused on the representation theory of general linear and symmetric groups (see [FH91, Mac95] for the basic theory). One of the guiding problems that accompanied the development of this project was the problem of computing equivariant resolutions (see Chapter 3). Even for easy (to define) rings like the homogeneous coordinate rings of Segre and Veronese varieties, equivariant resolutions are not currently known. For the latest developments in the subject, see [Sno10, EL11].

The SchurRings package implements the basic arithmetic operations in the representation rings of general linear and symmetric groups, or products of any combination of such. It also implements conversion routines between the standard e–, h–, p–, and s–bases (see Section 2), as well as various plethystic operations. Finally, the high point of the package is a routine meant to approximate equivariant resolutions of G–modules, for G a product of general linear and symmetric groups (Section 3).

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2. Symmetric Polynomials

Given a positive integer n and a field K, we consider the polynomial ring $S = K[x_1, \cdots, x_n]$. S has an action of the symmetric group S_n on n letters, given by the permutation of the variables. A symmetric polynomial in S is one that's invariant under the action of S_n.

There are four classes of symmetric polynomials that the package SchurRings implements:

(1) (elementary symmetric polynomials)
$$e_k(x_1, \cdots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}.$$

(2) (power–sum polynomials)
$$p_k(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^k.$$

(3) (complete symmetric polynomials)
$$h_k(x_1, \cdots, x_n) = \sum_{m \in M_k} m,$$
where M_k denotes the set of monomials of degree k in the variables $x_1, \cdots, x_n$.

(4) (Schur polynomials) Given a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ of k,
$$s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}.$$

Theorem 2.1 (Fundamental Theorem of Symmetric Polynomials, [Mac95, Section I.2])**.** *We let R_n denote $K[x_1, \cdots, x_n]^{S_n}$, the subring of S consisting of the collection of symmetric polynomials. We have*
$$R_n = K[e_1, \cdots, e_n] = K[h_1, \cdots, h_n].$$
If the characteristic of K is zero, then $R_n = K[p_1, \cdots, p_n]$.

In general, R_n has a basis (as a K-vector space) consisting of the polynomials $s_\lambda(x_1, \cdots, x_n)$, where $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition with at most n parts. If we write e_λ for the polynomial $\prod_i e_{\lambda_i}$, and likewise for h_λ and p_λ, we get that R_n has K-vector space bases $(e_\lambda)_\lambda$, $(h_\lambda)_\lambda$, and (in characteristic zero) $(p_\lambda)_\lambda$, indexed by partitions with parts of size at most n ($\lambda_i \le n$). One of the primary goals of the SchurRings package is to facilitate a fast transition between the e–, h–, p–, and s–bases.

Example 2.2. Below we consider the symmetric polynomial
$$e_3(x_1, x_2, x_3) - p_3(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3 - (x_1^3 + x_2^3 + x_3^3),$$
and express it in terms of the s–, e–, h–, and p–bases respectively.

```
i1 : loadPackage"SchurRings";
i2 : R = symmRing 3;
i3 : symFun = e_3 - p_3;
i4 : sFun = toS symFun
o4 = - s  + s
        3    2,1
```

```
i5 : eFun = toE sFun
        3
o5 = - e  + 3e e  - 2e
        1     1 2     3
i6 : hFun = toH eFun
o6 = h h  - 2h
      1 2     3
i7 : pFun = toP hFun
     1 3   1       2
o7 = -p  - -p p  - -p
     6 1   2 1 2   3 3
```

An important notion in the theory of symmetric functions and in representation theory is the notion of plethysm (see [Mac95, Section I.7]). This is an operation on symmetric polynomials that corresponds to the composition of Schur functors as GL–representations. It is often referred to as outer plethysm, as opposed to the operation of inner plethysm which corresponds to applying Schur functors to representations of a symmetric group. The SchurRings package can deal with both types of plethysm.

Example 2.3. We first compute the decomposition of $S_{2,1}(S_3(V))$ into a sum of irreducible representations, where V is a vector space of dimension 3, and S_λ denotes the Schur functor corresponding to the partition λ:

```
i8 : S = schurRing(s,3,GroupActing => "GL");
i9 : plethysm(s_{2,1},s_3)
o9 = s    + s    + s    + s      + s    + s      + s
      8,1    7,2    6,3    6,2,1    5,4    5,3,1    4,3,2
```

We can also decompose the third symmetric power of the standard representation of the symmetric group S_3:

```
i10 : T = schurRing(t,3,GroupActing => "Sn");
i11 : symmetricPower(3,t_{2,1})
o11 = t  + t    + t
       3    2,1    1,1,1
```

3. Equivariant resolutions

Given a group G and a finite dimensional G–representation W over a field K, we can form the symmetric algebra $S = \mathrm{Sym}(W)$. This is a graded ring whose degree d piece is $S_d = \mathrm{Sym}^d W$, the d–th symmetric power of the vector space W. S_d is a G-module, where the action is induced from the action of G on W. Fixing a basis $x_1, \cdots, x_n$ of W, S can be identified with the polynomial ring $K[x_1, \cdots, x_n]$. A (finitely generated, graded) S–module M is called G–equivariant if it admits an action of the group G which is compatible with the action of S on M. If this is the case, then M admits an equivariant resolution

$$F_\bullet : \quad F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n,$$

where

$$F_i = \bigoplus_{j\in\mathbb{Z}} B_{ij} \otimes S(-j),$$

B_{ij} are G–modules, and the differentials in $F_\bullet$ are minimal and respect the action of the group G. The dimensions β_{ij} of the vector spaces B_{ij} are the usual Betti numbers associated to the S–module M. We call B_{ij} the Betti modules of the equivariant module M.

One feature that the SchurRings package implements is the method schurResolution, which attempts to "guess" the Betti modules of a G–equivariant module M, where G is a product of a combination of general linear and symmetric groups. The assumption on which the "guessing" is based is that the differentials in the resolution $F_\bullet$ have maximal rank among all homomorphisms of G–modules between any successive terms F_i, F_{i+1}. Even though this assumption is not expected to be true generally, it does hold in many cases of interest: when M has a linear (or pure) resolution (e.g. Koszul complexes), or when M is the coordinate ring of some Segre or Veronese varieties, or the coordinate ring of some secant varieties of such.

Example 3.1. Let $G = S_3 \times GL(2)$, the product of the symmetric groups on three letters and the group of invertible transformations of a vector space V of dimension 2. Let U denote the permutation representation of S_3, and let $W = U \otimes V$. We let $S = \mathrm{Sym}(W)$ be as before, and let $M = K$ be the residue field of S. The resolution of M is known to be given by the Koszul complex [Eis95, Chapter 17].

```
i1 : loadPackage"SchurRings";
i2 : A = schurRing(a,3,GroupActing => "Sn");
i3 : B = schurRing(A,b,2,GroupActing => "GL");
i4 : rep = (a_3 + a_{2,1}) * b_1;
i5 : d = dim rep
o5 = 6
```

The symmetric functions corresponding to the characters of S_3 are linear combinations of a_λ, where λ is a partition of 3, while the characters of $GL(2)$ are combinations of b_λ, where λ is a partition with at most two parts. In particular, the character of U is $a_3 + a_{2,1}$, while that of V is b_1. Therefore the character of W is rep, the product of the two. As a G–representation, M is trivial and 1–dimensional, so its character is the product of the characters of the trivial representations of S_3 and $GL(2)$ respectively:

```
i6 : M = {a_3 * 1_B};
i7 : sR = schurResolution(rep,M,DegreeLimit => d)
o7 = {{(0, a b  )}, {(1, (a  + a   )b )}, {(2, (a    + a     )b
           3 ()           3    2,1  1           2,1    1,1,1  2
     ------------------------------------------------------------
     + (2a  + 2a   )b   )}, {(3, a     b  + (a  + 3a    +
          3     2,1  1,1           1,1,1 3     3     2,1
     ------------------------------------------------------------
     a     )b   )}, {(4, (a    + a     )b    + (2a  +
      1,1,1  2,1           2,1    1,1,1  3,1      3
     ------------------------------------------------------------
     2a   )b   )}, {(5, (a  + a   )b   )}, {(6, a b   )}}
       2,1  2,2           3    2,1  3,2           3 3,3
```

The Betti modules are returned as a list of lists of pairs, where each list of pairs corresponds to a term F_i in the equivariant resolution $F_\bullet$, and each pair consists of an integer j, and the character of the corresponding Betti module B_{ij}. In our case, the lists of pairs each consists of a single element, because the resolution is linear.

To check that we obtained the correct answer in this example, we need to verify that the Betti modules are exterior powers of W. We check this for $i = 3$:

```
i8 : sR#3
o8 = {(3, a      b  + (a  + 3a    + a      )b   )}
           1,1,1 3     3     2,1    1,1,1  2,1
i9 : exteriorPower(3,rep)
o9 = a      b  + (a  + 3a    + a      )b
      1,1,1 3     3     2,1    1,1,1  2,1
```

One caveat of our method of constructing equivariant resolutions is that it assumes "maximal cancellation" between the Betti modules, i.e. that there are no irreducible G–representations that are shared by any two modules $B_{i,j}$ and $B_{i+1,j}$. Nevertheless, our method yields a lower bound for the Betti modules, by detecting the syzygies whose presence is dictated by representation theory alone, independently of the ring structure of S and of the S–module structure of M.

Example 3.2. We end with the example of the equivariant resolution of the homogeneous coordinate ring of the cubic Veronese embedding of $\mathbb{P}^2$ (see also [Chi02, Section 2.6]):

```
i10 : S = schurRing(s,3);
i11 : rep = s_{3};
i12 : M = {1_S,s_{3},s_{6},s_{9},s_{12},s_{15},s_{18},
           s_{21},s_{24},s_{27}};
i13 : schurResolution(rep,M,DegreeLimit => 9)
o13 = {{(0, s  )}, {(2, s   )}, {(3, s      + s    + s      +
             ()          4,2          6,2,1    5,4    5,3,1
      -----------------------------------------------------------
      s     )}, {(4, s      + s      + s      + s      + s      +
       4,3,2          7,4,1    7,3,2    6,5,1    6,4,2    6,3,3
      -----------------------------------------------------------
      s      + s     )}, {(5, s      + s      + s      + s      +
       5,5,2    5,4,3          8,5,2    8,4,3    7,6,2    7,5,3
      -----------------------------------------------------------
      s      + s      + s     )}, {(6, s      + s      + s      +
       7,4,4    6,6,3    6,5,4          9,5,4    8,7,3    8,6,4
      -----------------------------------------------------------
      s     )}, {(7, s     )}, {(9, s     )}}
       7,6,5          9,7,5          9,9,9
```

The shape of the resolution suggests that the homogeneous coordinate ring of the cubic Veronese surface is Gorenstein, which is indeed the case, as can be checked by a quick computation of its Hilbert series.

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