COMPUTING SCHUR COMPLEXES

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ABSTRACT. We describe a Macaulay2 package for computing Schur complexes. This package expands on the `ChainComplexOperations` package by David Eisenbud.

1. INTRODUCTION

Let R be a commutative ring. The goal of this article is to describe the Macaulay2 package `SchurComplexes`, which computes the Schur complex $\mathbf{S}_\lambda(F)$ associated to a bounded complex F of finitely generated free R-modules and a partition λ.

Schur complexes are a simultaneous generalization of the symmetric and exterior power operations on complexes. The notion of a Schur complex was introduced by Nielsen in [Nie78] in the characteristic 0 setting, and it was generalized to the characteristic free setting by Akin–Buchsbaum–Weyman in [ABW82].

The importance of such operations on complexes is illustrated by Walker's recent proof of the weak Buchsbaum–Eisenbud–Horrocks conjecture [Wal17, Theorem 2.4] in which exterior and symmetric squares of complexes play a crucial role. Walker's breakthrough work led to Eisenbud's implementation of the second exterior and symmetric power for complexes in the Macaulay2 package `ChainComplexOperations`. The work suggests that properties of Schur complexes should be further developed. Our goal here is to expand on this package by implementing the construction of an arbitrary Schur complex.

In Section 2, we provide some background on Schur complexes, following the detailed treatment in Weyman's book [Wey03, Section 2.4]. In particular, we recall the "straightening algorithm" of [ABW82] which expresses a $\mathbb{Z}/2$-graded tableau as a $\mathbb{Z}$-linear combination of so-called "standard" $\mathbb{Z}/2$-graded tableaux; the implementation of this algorithm is the key component of the `SchurComplexes` package. Section 3 contains some examples of computations using `SchurComplexes`.

2. BACKGROUND ON SCHUR COMPLEXES

This section closely follows [Wey03, Section 2.4]. Let $F = (F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d)$ be a bounded complex of finitely generated free R-modules. Denote by F_{even} (resp. F_{odd}) the direct sum of the even (resp. odd) degree components of F, and choose bases $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ of F_{odd} and F_{even}, respectively, which are unions of bases of the F_i.

2.1. **Exterior powers of complexes.** Fix a positive integer r, and let $T^r(F)$ denote the rth tensor power of F. $T^r(F)$ may be equipped with a Σ_r-action in the following way:

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_r) = \pm x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(r)},$$

where the x_i are homogeneous elements of F, and the sign is determined by declaring that transposing the elements x_i and x_j contributes the sign $(-1)^{|x_i||x_j|}$.

Definition 2.1. The rth exterior power $\bigwedge^r F$ is defined to be the subcomplex of $T^r(F)$ consisting of antisymmetric tensors, i.e. elements $x \in T^r(F)$ such that $\sigma \cdot x = (-1)^{\text{sign}(\sigma)}x$ for all x.

There is a canonical R-linear embedding

$$\iota \colon \bigoplus_i D_i(F_{\text{odd}}) \otimes \bigwedge^{r-i}(F_{\text{even}}) \hookrightarrow T^r(F)$$

whose image is precisely $\bigwedge^r F$. Here, $D_i(F_{\text{odd}}) := \text{Sym}_i((F_{\text{odd}})^*)^*$, the ith divided power of F_{odd}. We now describe this embedding in detail.

Let $\varepsilon \colon T(F) \to T(F)$ denote the anti-symmetrization map

$$(x_1 \otimes \cdots \otimes x_i) \mapsto \sum_{\sigma \in \Sigma_i} (-1)^{\text{sign}(\sigma)}\sigma \cdot (x_1 \otimes \cdots \otimes x_i).$$

The composition $T(F_{\text{even}}) \hookrightarrow T(F) \xrightarrow{\varepsilon} T(F)$ factors through $\bigwedge F_{\text{even}}$, inducing an embedding

$$\iota_\Lambda \colon \bigwedge F_{\text{even}} \hookrightarrow T(F).$$

By [Rob63] Proposition IV.5, there is an embedding of R-algebras

$$\iota_D \colon D(F_{\text{odd}}) \to T(F_{\text{odd}})$$

such that $\iota_D(x^{(j)}) = \underbrace{x \otimes \cdots \otimes x}_{j \text{ copies}}$ for all $x \in F_{\text{odd}}$, where the target is equipped with the shuffle product. The embedding ι is defined on each summand $D_i(F_{\text{odd}}) \otimes \bigwedge^{r-i} F_{\text{even}}$ by

$$\sum_{\sigma \in \Sigma_{i,r-i}} (-1)^{\text{sign}(\sigma)}\sigma \cdot (\iota_D \otimes \iota_\Lambda),$$

where $\Sigma_{i,r-i} \subseteq \Sigma_r$ denotes the set of $(i, r-i)$ shuffles.

Example 2.2. Let $x, y \in R$, and take F to be the Koszul complex

$$R \xleftarrow{(x \quad y)} R^{\oplus 2} \xleftarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R$$

on x and y, lying in homological degrees $0, 1$, and 2. Then $\bigwedge^2(F)$ is the complex

$$R^{\oplus 2} \xleftarrow{\begin{pmatrix} y & x & 0 & x \\ 0 & y & x & -y \end{pmatrix}} R^{\oplus 4} \xleftarrow{\begin{pmatrix} 2x & 0 \\ -y & x \\ 0 & -2y \\ -y & -x \end{pmatrix}} R^{\oplus 2},$$

lying in homological degrees $1, 2$, and 3.

2

The complex $\bigwedge F := \bigoplus_{r\geq 0} \bigwedge^r F$ is equipped with a product
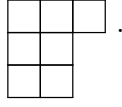
$$\mu\colon \overset{r_1}{\bigwedge} F \otimes \overset{r_2}{\bigwedge} F \to \overset{r_1+r_2}{\bigwedge} F$$

and a coproduct

$$\Delta\colon \overset{r}{\bigwedge} F \to \bigoplus_{r_1+r_2=r} \overset{r_1}{\bigwedge} F \otimes \overset{r_2}{\bigwedge} F.$$

For explicit formulas for μ and Δ, we refer the reader to the proof of [Wey03, Proposition 2.4.1].

2.2. **Schur complexes.** Let r be a positive integer, and let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of r, where $\lambda_i \geq \lambda_{i+1}$. We will encode partitions with Young diagrams. For example, the partition $(3, 2, 2)$ of 7 corresponds to the diagram

Let $c_1, \ldots, c_t$ denote the lengths of the columns of λ.

Definition 2.3. The *Schur complex* $\mathbf{S}_\lambda(F)$ is the quotient $(\bigwedge^{c_1} F \otimes \cdots \otimes \bigwedge^{c_t} F)/R$, where R is the sum of submodules

$$\overset{c_1}{\bigwedge} F \otimes \cdots \otimes \overset{c_{a-1}}{\bigwedge} F \otimes R_{a,a+1} \otimes \overset{c_{a+2}}{\bigwedge} F \otimes \cdots \otimes \overset{c_t}{\bigwedge} F.$$

Here, $R_{a,a+1}$ is the submodule spanned by the images of the compositions

$$\Theta(a, u, v; F)\colon \overset{u}{\bigwedge} F \otimes \overset{c_a-u+c_{a+1}-v}{\bigwedge} F \otimes \overset{v}{\bigwedge} F \xrightarrow{1\otimes\Delta\otimes1} \overset{u}{\bigwedge} F \otimes \overset{c_a-u}{\bigwedge} F \otimes \overset{c_{a+1}-v}{\bigwedge} F \otimes \overset{v}{\bigwedge} F$$

$$\xrightarrow{\mu\otimes\mu} \overset{c_a}{\bigwedge} F \otimes \overset{c_{a+1}}{\bigwedge} F$$

for $u + v < c_{a+1}$.

Remark 2.4. Our definition of the Schur complex differs from the one in [Wey03, Section 2.4]: Weyman's definition of the Schur complex of F with respect to λ is equal to $\mathbf{S}_{\lambda^*}(F)$, where λ^* is the conjugate of λ, i.e. the result of transposing the rows and columns of λ.

Example 2.5. Of course, if $\lambda = (1, \ldots, 1)$, $\mathbf{S}_\lambda(F) = \bigwedge^r F$. If $\lambda = (r)$, $\mathbf{S}_\lambda(F) = \mathrm{Sym}_r(F)$ (see [Wey03, Section 2.4] for the definition of the symmetric power of a complex).

The basis of F chosen above determines a basis for each $\bigwedge^{c_i} F$: namely, the images under ι of elements of the form

$$\{e_1^{(j_1)} \cdots e_m^{(j_l)} \otimes f_{k_1} \wedge \cdots \wedge f_{k_{c_i-(j_1+\cdots+j_l)}}\}$$

(from now on, we will tacitly identity these basis elements with their images under ι). These bases combine to form an R-linear spanning set $\mathcal{S}$ for $\mathbf{S}_\lambda(F)$. We will write elements of $\mathcal{S}$ as $\mathbb{Z}/2$-*graded Young tableaux of shape* λ, i.e. functions

$$T\colon \{1, \ldots, r\} \to \{-m, \ldots, -1\} \cup \{1, \ldots, n\},$$

where, as above, $m = \mathrm{rank}(F_{\mathrm{odd}})$ and $n = \mathrm{rank}(F_{\mathrm{even}})$. Here, divided power factors correspond to negative values, and exterior factors correspond to positive values. For instance,

if $\lambda = (3,3,1)$ and $m = 2 = n$, the element $(e_2^{(2)} \otimes f_1) \otimes (e_1 \otimes f_1) \otimes (f_1 \wedge f_2)$ in $\mathbf{S}_{(3,3,1)}(F)$ corresponds to the function

l	1	2	3	4	5	6	7
$T(l)$	-2	-2	1	-1	1	1	2

which we express as the Young tableau

$$
T = \begin{array}{|c|c|c|}
\hline
-2 & -1 & 1 \\
\hline
-2 & 1 & 2 \\
\hline
1 \\
\cline{1-1}
\end{array} .
$$

We will call such a tableau *standard* if

(A) the columns increase from top to bottom, with equality possible only for negative values, and

(B) the rows increase from left to right, with equality possible only for positive values.

Remark 2.6. Our definition of a standard tableau is the transpose of Weyman's in [Wey03, Definition 1.1.12(c)] (cf. Remark 2.4 above).

For instance, the tableau T above is standard. The tableaux

$$
\begin{array}{|c|c|c|}
\hline
-2 & -1 & 1 \\
\hline
-2 & -1 & 2 \\
\hline
-3 \\
\cline{1-1}
\end{array}
,\qquad
\begin{array}{|c|c|c|}
\hline
-2 & -1 & 1 \\
\hline
1 & 1 & 2 \\
\hline
1 \\
\cline{1-1}
\end{array}
,\qquad
\begin{array}{|c|c|c|}
\hline
-2 & -1 & -1 \\
\hline
-2 & 1 & 2 \\
\hline
1 \\
\cline{1-1}
\end{array}
$$

are non-standard.

Proposition 2.7 ([Wey03, Proposition 2.4.2]). *The standard $\mathbb{Z}/2$-graded tableaux of shape λ form an R-linear basis of $\mathbf{S}_\lambda(F)$.*

We compute the differentials in $\mathbf{S}_\lambda(F)$ with respect to this basis in the `SchurComplexes` package. It is therefore essential for us to implement an algorithm for writing the image of a standard tableau under the differential in $\mathbf{S}_\lambda(F)$ as a linear combination of standard tableaux. The proof of [Wey03, Proposition 2.4.2] explains such an algorithm: the "straightening algorithm" of [ABW82]. We now discuss this algorithm in detail.

2.3. **The straightening algorithm.** As in the previous subsection, $\lambda = (\lambda_1, \ldots, \lambda_s)$ denotes a partition of a positive integer r with column lengths $c_1, \ldots, c_t$. The straightening algorithm is a process for writing a $\mathbb{Z}/2$-graded tableau in the spanning set $\mathcal{S}$ of $\mathbf{S}_\lambda(F)$ described above as a $\mathbb{Z}$-linear combination of standard tableaux. Here is how it works.

Let T be a tableau in $\mathcal{S}$. We make the following observations:

- If a column of T contains a repeated positive entry, $T = 0$ in $\mathbf{S}_\lambda(F)$.
- Since the divided power (resp. exterior) algebra is commutative (resp. skew commutative), rearranging the columns in T so that it satisfies (A) only changes the element of $\mathbf{S}_\lambda(F)$ represented by T up to a sign.

With these facts in mind, we recall the straightening algorithm:

4

- **Input**: A tableau $T \in \mathcal{S}$.

- **Step 1:** Denote by T' the result of rearranging the columns of T so that they satisfy (A), and let $\sigma \in \{\pm 1\}$ denote the resulting sign.

- **Step 2:** If T' satisfies (B), output $\sigma T'$. Otherwise, choose the topmost row, say the wth row, with a "violation" of (B). Let $T'(i, j)$ denote the entry with horizontal coordinate i and vertical coordinate j in the Young diagram, counting from the top-left corner. So $T'(i, j)$ is the entry in the ith column and the jth row. Choose the smallest index a such that either $T'(a, w) > T'(a+1, w)$ or $T'(a, w) = T'(a+1, w) < 0$. Then, choose the smallest index w' such that $T'(a + 1, w) < T'(a + 1, w' + 1)$; if no such index exists, set $w' = c_{a+1}$.

- **Step 3:** Set $u := w - 1$ and $v := c_{a+1} - w'$. Define
  (1) $V_1 \in \bigwedge^u F$ to be the element corresponding to the first u entries in the ath column of T',
  (2) $V_2 \in \bigwedge^{c_a - u + c_{a+1} - v} F$ to be the element corresponding to the last $c_a - u$ entries in the ath column of T' followed by the first $c_{a+1} - v = w'$ entries of the $(a + 1)^{\text{st}}$ column of T',
  (3) $V_3 \in \bigwedge^v F$ to be the element corresponding to the last v entries in the $(a + 1)^{\text{st}}$ column of T'.
  For $k \in \{1, \ldots, a - 1, a + 2, \ldots, s\}$, define U_k to be the element of $\bigwedge^{c_k} F$ which corresponds to the kth column of T'. Recall that

  $$L := (1 \otimes \Theta(a, u, v; F) \otimes 1)(U_1 \otimes \cdots \otimes U_{a-1} \otimes V_1 \otimes V_2 \otimes V_3 \otimes U_{a+2} \otimes \cdots \otimes U_s)$$

  is 0 in $\mathbf{S}_\lambda(F)$, where $\Theta(a, u, v; F)$ is as in Definition 2.3. L is a $\mathbb{Z}$-linear combination of tableaux in $\mathcal{S}$ which contains T' with coefficient 1. If each tableau in the sum $T' - L$ is standard, output $\sigma(T' - L)$. Otherwise, repeat this algorithm on each tableaux in $\sigma(T' - L)$, keeping track of the coefficients.

The key observation is that each tableau appearing in the linear combination $T' - L$ from Step 3 is strictly "smaller" than T', in the sense described in [Wey03, Section 1.1], and so the algorithm does indeed terminate.

Example 2.8. Let's apply the straightening algorithm to the tableau

$$T = \begin{array}{|c|c|c|} \hline -3 & 2 & -1 \\ \hline -2 & 1 & 3 \\ \hline -2 & 3 \\ \cline{1-2} \end{array}.$$

- **Step 1:** The middle column needs to be rearranged. Since $f_2 f_1 f_3 = -f_1 f_2 f_3$ in $\bigwedge^3 F_{\text{odd}}$, we have

$$T' = \begin{array}{|c|c|c|} \hline -3 & 1 & -1 \\ \hline -2 & 2 & 3 \\ \hline -2 & 3 \\ \cline{1-2} \end{array}$$

and $\sigma = -1$.

- **Step 2:** T' is not standard. Here, $w = 1$, $a = 2$, and $w' = 1$.

- **Step 3:** Here, $u = 0$ and $v = 1$. We have $U_1 = e_3 e_2^{(2)}$, $V_2 = e_1 \otimes f_1 \wedge f_2 \wedge f_3$, and $V_3 = f_3$ (since $u = 0$, V_1 plays no role), so

$$L = (1 \otimes \Theta(2, 0, 1; F))(e_3 e_2^{(2)} \otimes (e_1 \otimes f_1 \wedge f_2 \wedge f_3) \otimes f_3).$$

In this case, $\Theta(2, 0, 1)$ is the composition

$$(\bigwedge^4 F) \otimes F \xrightarrow{\Delta \otimes 1} (\bigwedge^3 F \otimes F) \otimes F \xrightarrow{\mathrm{id} \otimes \mu} \bigwedge^3 F \otimes \bigwedge^2 F.$$

By the proof of [Wey03, Proposition 2.4.1(a)], the relevant component of the coproduct

$$\Delta \colon \bigwedge^4 F \to \bigwedge^3 F \otimes F$$

in $\Theta(2, 0, 1; F)$ is:

$$D_1(F_{\mathrm{odd}}) \otimes \bigwedge^3 F_{\mathrm{even}} \xrightarrow{\Delta_D \otimes \Delta_\Lambda} \bigoplus_{i=0}^1 D_i(F_{\mathrm{odd}}) \otimes D_{1-i}(F_{\mathrm{odd}}) \otimes \bigwedge^{3-i} F_{\mathrm{even}} \otimes \bigwedge^i F_{\mathrm{even}}$$

$$\xrightarrow{\tau} \bigoplus_{i=0}^1 D_i(F_{\mathrm{odd}}) \otimes \bigwedge^{3-i} F_{\mathrm{even}} \otimes D_{1-i}(F_{\mathrm{odd}}) \otimes \bigwedge^i F_{\mathrm{even}},$$

where Δ_D and Δ_Λ are the coproducts on the ordinary divided power and exterior algebra, and τ transposes the middle factors and multiplies by the sign $(-1)^{(1-i)(3-i)}$.

Note: one might think that, since the elements of $\bigwedge^{3-i} F_{\mathrm{even}}$ have even degree, transposing the middle two factors should not introduce a sign. But recall that we are considering $D_{1-i}(F_{\mathrm{odd}}) \otimes \bigwedge^{3-i} F_{\mathrm{even}}$ as a submodule of $T^4(F)$ via the embedding ι, which shuffles together the elements of $D_{1-i}(F_{\mathrm{odd}})$ and $\bigwedge^{3-i} F_{\mathrm{even}}$: this is why it is necessary to multiply by $(-1)^{(1-i)(3-i)}$. The general rule here is: when one transposes the factors of $D_s(F_{\mathrm{odd}}) \otimes \bigwedge^t F_{\mathrm{even}}$, one must introduce the sign $(-1)^{st} = \mathrm{sign}((1\ 2 \cdots s + t)^t)$.

Applying the formula for Δ_Λ in [Wey03, Section 1.1], one gets

$$L = -(e_3 e_2^{(2)}) \otimes (f_1 \wedge f_2 \wedge f_3) \otimes (e_1 \otimes f_3) - (e_3 e_2^{(2)}) \otimes (e_1 \otimes f_1 \wedge f_3) \otimes (f_2 \wedge f_3)$$
$$+ (e_3 e_2^{(2)}) \otimes (e_1 \otimes f_2 \wedge f_3) \otimes (f_1 \wedge f_3),$$

and therefore

$$T = \begin{array}{|c|c|c|}
\hline
-3 & -1 & 2 \\
\hline
-2 & 1 & 3 \\
\hline
-2 & 3 \\
\cline{1-2}
\end{array} - \begin{array}{|c|c|c|}
\hline
-3 & -1 & 1 \\
\hline
-2 & 2 & 3 \\
\hline
-2 & 3 \\
\cline{1-2}
\end{array}.$$

Both of these tableaux are standard, so we're done.

3. Examples of computations using the SchurComplexes package

The `SchurComplexes` package has two main functions:

- `straightenTableau`, which applies the straightening algorithm to a tableau, and
- `schurComplex`, which computes the Schur complex of a bounded complex of finitely generated free modules.

3.1. Using the `straightenTableau` function.

Let's apply `straightenTableau` to the tableau in Example 2.8. First, we load the package:

```
i1 : loadPackage "SchurComplexes.m2"
```

We encode the tableau T in a `HashTable`:

```
i2 : T = new HashTable from {(1,1) => -3, (1,2) => -2, (1,3) => -2,
(2,1) => 2, (2,2) => 1, (2,3) => 3, (3,1) => -1, (3,2) => 3}
```

We encode the partition $(3, 3, 2)$ in a `List`:

```
i3 : lambda = {3,3,2}
```

Now, we apply `straightenTableau` to the pair (T, lambda):

```
i4 : straightenTableau(T, lambda)

o4 = HashTable{HashTable{(1, 1) => -3} => 1 }
                        (1, 2) => -2
                        (1, 3) => -2
                        (2, 1) => -1
                        (2, 2) => 1
                        (2, 3) => 3
                        (3, 1) => 2
                        (3, 2) => 3
              HashTable{(1, 1) => -3} => -1
                        (1, 2) => -2
                        (1, 3) => -2
                        (2, 1) => -1
                        (2, 2) => 2
                        (2, 3) => 3
                        (3, 1) => 1
                        (3, 2) => 3
```

The output is a `HashTable` which assigns a coefficient to each standard tableau in the linear combination comprising the straightening of T. Notice that the output agrees with the calculation in Example 2.8.

3.2. Using the `schurComplex` function.

Let $R = \mathbb{Q}[x_{i,j}]$, where $1 \le i \le 2$ and $1 \le j \le 4$, and let

$$F\colon R^{\oplus 4} \xrightarrow{(x_{i,j})} R^{\oplus 2}$$

denote the generic 2×4 matrix, considered as a complex concentrated in degrees 1 and 0. By [Wey03, Exercise 6.34(d)], $\mathbf{S}_{(3)}(F) = \mathrm{Sym}_3(F)$ has nonzero homology only in degree 0. We now use the `schurComplex` method to compute $\mathbf{S}_{(3)}(F)$ and verify this fact.

We first load the package, fix our ground ring R, and define our complex F:

```
i1 : loadPackage "SchurComplexes.m2"
i2 : R = QQ[x11,x21,x12,x22,x13,x23,x14,x24];
i3 : M = genericMatrix(R,x11,2,4);
i4 : F = new ChainComplex;
i5 : F.ring = R; F#0 = target M; F#1 = source M; F.dd#1 = M;
```

The method `schurComplex` takes as input a `ChainComplex` and a `List` which encodes the partition. Let's define our partition and compute $\mathbf{S}_{(3)}(F)$:

```
i6 : lambda = {3};
i7 : S = schurComplex(lambda,F)

        4        12        12        4
o7 = R   <-- R     <-- R     <-- R

       0        1         2        3
```

Finally, let's check that $\mathbf{S}_{(3)}(F)$ has trivial homology in degrees greater than 0:

```
i8 : apply((length S)+1,i->reduceHilbert hilbertSeries HH_i(S))

            4      0  0  0
o8 = {--------, -, -, -}
            5    1  1  1
       (1 - T)
```

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