The elementary socle series: Problems and Questions

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Abstract: This is a talk about a group of problems that has obsessed me for well over 20 years (cf. Kucera[2008]). It comes in versions for an "algebra audience" and for a "model theory" audience.

The *elementary socle series* of a module was introduced by Ivo Herzog [1993] as a natural first-order definable analogue of the algebraic socle series (*Loewy series*). If N is a totally transcendental (tt) module, then the elementary socle series *exhausts* N, that is, for some ordinal α, $\mathrm{soc}^{\alpha}(N) = N$.

An important family of examples of such modules are injective (left) modules E over a (left) noetherian ring R. If R is commutative, the elementary socle series of an indecomposable injective E corresponds to the well-known hierarchy of Matlis [1958]. If R is not commutative, neither the Loewy series nor the more generally applicable *fundamental series* of Jategaonkar [1986] always exhausts E. Because the elementary socle series must exhaust E, it might provide a route of attack on many difficult unsolved problems in non-commutative algebra.

I will start with a very brief overview of the model theory of modules and in particular the meaning of "definable" in this context, and provide a few details of the two motivating algebraic contexts just mentioned.

We know that each term of the elementary socle series of an indecomposable tt module N is a definably closed, fully invariant, submodule of N: and not much else. A hasty sketch of the proof will show some of the ways model-theoretic and algebraic ideas interact.

I will finish by outlining three main questions about the structure of the elementary socle series and some associated side problems. In particular, I do not know an example where $N/\mathrm{soc}^{\alpha}(N)$ is NOT tt. If it were always tt, this would lead to a possible method of attack on the structure problems.