BOIJ-SÖDERBERG THEORY FOR NON-STANDARD GRADED RINGS

BRETT BARWICK, JENNIFER BIERMANN, DAVID COOK II, W. FRANK MOORE, CLAUDIU RAICU, AND DUMITRU STAMATE

Abstract. In this note we investigate the cone of Betti tables of graded modules over a non-standard graded polynomial ring.

1. Introduction

Boij and Söderberg conjectured a beautiful structure theorem on the cone of Betti tables of graded modules over the polynomial ring $R = k[x_1, \ldots, x_n]$ where $\deg x_i = 1$. The primary objects of interest in this theory are the extremal rays of this cone, which are provided by the Betti diagrams of *pure* modules; a module is pure if each of its syzygies is generated in a unique degree. The motivation for this conjecture was the multiplicity conjecture [3] of Herzog-Huneke-Srinivasan, which was known to hold in the pure case by the Herzog-Kühl equations [5]. Therefore, a natural question arose: Is every Betti diagram (up to scaling) a positive rational linear combination of pure Betti tables? The existence of pure diagrams were proved in characteristic zero by Eisenbud-Fløystad-Weyman [1], and the full conjecture was proven in arbitrary characteristic by Eisenbud-Schreyer [2] by providing a connection between Betti tables of modules over R and cohomology tables of vector bundles over $\mathbb{P}^{n-1}$.

Our group originally wanted to focus on describing the cone of Betti diagrams of graded modules over $R = k[x_1, ..., x_n]$ where $\deg x_i = e_i$. We discovered early on that this was quite a lofty goal, and so we restricted soon to the case of $R = k[x, y]$ with $\deg x = 1$ and $\deg y = 2$. There are some general reductions that one can make in this case:

2. Herzog-Kühl equations

Suppose $R = k[x_1, \ldots, x_n]$ where $\deg x_i = d_i$. Let M be an Artinian graded R-module with a minimal free resolution of the form

$$0 \to F_p \to \cdots \to F_0 \to M \to 0$$

where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$. Then the $\beta_{i,j}$ are the *Betti numbers* of M. In this case, the Hilbert series of M is given by

$$H_M(t) = \frac{\sum_{i,j} (-1)^i \beta_{i,j} t^j}{\prod_{i=1}^n (1 - t^{d_i})}.$$

As the Hilbert series of an Artinian module is a polynomial, we see that

$$\prod_{i=1}^{n} (1 - t^{d_i}) \;\Big|\; \sum_{i,j} (-1)^i \beta_{i,j} t^j$$

and hence there exist $\sum_{i=1}^n d_i$ equations on the Betti numbers of M which must be satisfied.

Specializing to the case $R = k[x, y]$ with $\deg x = 1, \deg y = 2$, we have the following three equations:

$$\sum_{i,j} (-1)^i \beta_{i,j} = 0, \ \ \sum_{i,j} (-1)^i j \beta_{i,j} = 0, \text{ and } \sum_{i,j} (-1)^{i+j} \beta_{i,j} = 0.$$

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3. Pure diagrams

Let $(d_0 < d_1 < d_2)$ be a degree sequence. Without loss of generality, we may assume that $d_0 = 0$. If d_1 and d_2 are even, then one can realize the diagram $\begin{pmatrix} * & - & - \\ \vdots & \vdots & \vdots \\ - & * & - \\ \vdots & \vdots & \vdots \\ - & - & * \end{pmatrix}$ by constructing the module corresponding to the pure Betti diagram for the degree sequence $(0, \frac{d_1}{2}, \frac{d_2}{2})$ as in Eisenbud-Schreyer [2], but using the ring $k[x^2, y^2]$ (with both x and y of degree 1), and tensoring the module up along the inclusion $k[x^2, y^2] \subseteq k[x, y^2]$.

Suppose that $(0, d_1, d_2)$ is the degree sequence of a pure Artinian module. The Herzog-Kühl equations become

$$\begin{aligned} \beta_{0,0} - \beta_{1,d_1} + \beta_{2,d_2} &= 0 \\ -d_1\beta_{1,d_1} + d_2\beta_{2,d_2} &= 0 \\ \beta_{0,0} - (-1)^{d_1}\beta_{1,d_1} + (-1)^{d_2}\beta_{2,d_2} &= 0 \end{aligned}$$

Subtracting the first and the last equation shows that

$$\left(1 - (-1)^{d_1}\right)\beta_{1,d_1} = \left(1 - (-1)^{d_2}\right)\beta_{2,d_2},$$

which implies that if one of d_1 or d_2 is odd, then so is the other, since β_{i,d_i} are positive. In this case, $\beta_{1,d_1} = \beta_{2,d_2}$, in which case the first equation above implies $\beta_{0,0}$ is zero. Therefore, both d_1 and d_2 are even, and we described how to achieve such Betti tables with modules above.

4. Quasi-pure diagrams

Definition 4.1. *Given a degree sequence $d = (d_0 < d'_1 < d''_1 < d_2)$, we define a quasi-pure diagram of type d to be one for which*

$$\beta_{i,j} = 0 \text{ whenever } (i,j) \notin \{(0,d_0), (1,d'_1), (1,d''_1), (2,d_2)\}.$$

Given a quasi-pure diagram corresponding to a module of finite length, its Betti numbers must satisfy the Herzog-Kühl equations

$$\begin{gathered} \beta_{0,d_0} + \beta_{2,d_2} = \beta_{1,d'_1} + \beta_{1,d''_1}, \\ (-1)^{d_0}\beta_{0,d_0} + (-1)^{1+d'_1}\beta_{1,d'_1} + (-1)^{1+d''_1}\beta_{1,d''_1} + (-1)^{2+d_2}\beta_{2,d_2} = 0, \\ (-1)^{d_0}d_0\beta_{0,d_0} + (-1)^{d'_1}d'_1\beta_{1,d'_1} + (-1)^{d''_1}d''_1\beta_{1,d''_1} + (-1)^{d_2}d_2\beta_{2,d_2} = 0. \end{gathered}$$

We can shift the degrees of any quasi-pure diagram with degree sequence d to make $d_0 = 0$, thus we may assume for simplicity that this is the case for all our diagrams. We distinguish two cases, according to the parity of d_2.

Case 1: d_2 even

The Herzog-Kühl equations together with the nonnegativity of the Betti numbers imply that the only nonzero entries in the middle column of a quasi-pure diagram β may show up only on odd rows, that is

$$\beta_{1,i} = 0 \text{ for } i \text{ even}.$$

It follows easily that either β is pure, or it decomposes as the sum of two pure diagrams.

Case 2: d_2 odd

In this case, the Herzog-Kühl equations determine the entries of the Betti diagram β uniquely up to a constant factor. To see this, let

$$a = \beta_{0,0}, \quad b = \beta_{1,d'_1}, \quad z = \beta_{1,d''_1}, \quad t = \beta_{2,d_2}.$$

The first two Herzog-Kühl equations yield

$$\begin{aligned} a + t &= b + z \\ a - t &= \pm b \pm z \end{aligned}$$

according to the parity of d_1', d_1''. If both signs are the same (equivalently if d_1', d_1'' have the same parity) it follows that either $a = 0$ or $t = 0$, so β can't be the Betti diagram of a finite length module. We may therefore assume that d_1' and d_1'' have distinct parity. We distinguish two subcases:

2.1: d_1' even, d_1'' odd. The two equations above yield

$$a = b, \quad z = t$$

and the 3rd Herzog-Kühl equation becomes

$$d_1'y + d_1''z = d_2 t$$

which combined with the previous relations implies

$$\begin{aligned} a &= b = \lambda \cdot (d_2 - d_1''), \\ z &= t = \lambda \cdot d_1', \end{aligned}$$

for some constant λ.

2.2: d_1' odd, d_1'' even. An analogous calculation shows that in this case we must have

$$\begin{aligned} a &= z = \lambda \cdot (d_2 - d_1'), \\ b &= t = \lambda \cdot d_1'', \end{aligned}$$

for some λ.

5. Powers of the maximal ideal and variations

One of the classes of Betti diagrams which we conjecture to be extremal are the Betti diagrams of the powers of the maximal ideal, $\mathfrak{m}$. These diagrams have the following form:

$$\beta(R/\mathfrak{m}^n) = \begin{pmatrix} 1 & - & - \\ - & - & - \\ \vdots & \vdots & \vdots \\ - & - & - \\ - & 1 & - \\ - & 1 & 1 \\ \vdots & \vdots & \vdots \\ - & 1 & 1 \end{pmatrix}$$

where there are $n-2$ rows of zeros after the first row and the last n rows are of the form $\begin{pmatrix} - & 1 & 1 \end{pmatrix}$. By taking powers of the maximal ideal and replacing x^n with a higher power of x or replacing y^n with a higher power of y, we obtain two other classes of modules whose Betti diagrams may be pure in many cases. More precisely define two classes of modules for $n, r \geq 1$ as follows

$$\begin{aligned} A_{n,r} &= R/(x^{n+r}, x^{n-1}y, x^{n-2}x^2, \ldots, xy^{n-1}, y^n) \\ B_{n,r} &= R/(x^n, x^{n-1}y, x^{n-2}x^2, \ldots, xy^{n-1}, y^{n+r}) \,. \end{aligned}$$

The effect of increasing the exponent of x^n on the Betti diagram of R/m^n is to shift the Betti numbers $\beta_{1,n}$ and $\beta_{2,n+2}$ down r rows in the diagram. On the other hand, increasing the exponent of y^n adds rows of zeros before the last non-zero row of the diagram. The Betti diagram of $B_{n,r}$ has $2r$ rows of zeros before the last non-zero row. So for example,

$$\beta(A_{3,1}) = \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & - & - \\ - & 2 & - \\ - & 1 & 2 \\ - & 1 & 1 \end{pmatrix}, \quad \beta(B_{3,1}) = \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 1 & - \\ - & 1 & 1 \\ - & 1 & 1 \\ - & - & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}.$$

Some of these Betti diagrams are decomposable into types of diagrams which we conjecture to be pure. For example,

$$\beta(A_{2,2}) = \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 1 & - \\ - & 2 & 1 \\ - & - & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 2 & - \\ - & 1 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & - & - \\ - & - & - \\ - & - & - \\ - & 3 & - \\ - & - & 2 \end{pmatrix}$$

Others, however, appear to be indecomposable.

6. A Basic Case

Our approach was to include all the extremal rays of the above form for a particular degree range, compute the facet equations of the cone defined by these rays, and see if all Betti tables in this degree range satisfy these inequalities. The first case we looked at carefully is the cone of Betti tables of minimal resolutions of Artinian modules having degree bounds $(0, 1, 3)$ to $(0, 4, 5)$. Note that since there is only one variable of degree 1, the smallest degree of a second syzygy is 3. This cone has the 'shape' $\begin{pmatrix} * & * & - \\ - & * & * \\ - & * & * \\ - & * & * \end{pmatrix}$.

The following are the candidates for extremal rays coming from the above remarks in this case:

$$\begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \\ - & - & - \\ - & - & - \end{pmatrix}, \begin{pmatrix} 2 & 1 & - \\ - & 2 & - \\ - & - & - \\ - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & - \\ - & - & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 2 & - \\ - & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \\ - & - & - \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 1 & 1 \\ - & 1 & 1 \end{pmatrix}.$$

These Betti diagrams are realized by the cokernels of the following matrices respectively:

$$\begin{pmatrix} x & y \end{pmatrix}, \begin{pmatrix} y & x^2 & 0 \\ 0 & -y & x \end{pmatrix}, \begin{pmatrix} x^3 & y \end{pmatrix}, \begin{pmatrix} x & y^2 \end{pmatrix}, \begin{pmatrix} x^3 & xy & y^2 \end{pmatrix}, \begin{pmatrix} x^2 & y \end{pmatrix}, \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}.$$

Using the `FourierMotzkin` package in Macaulay2 [4], one obtains the equations defining the supporting hyperplanes of the cone defined by these rays:

$$\begin{aligned} \beta_{2,4} - \beta_{1,3} + 2\beta_{2,5} - 2\beta_{1,4} &\geq 0 \\ \beta_{2,5} - \beta_{1,3} &\geq 0 \\ \beta_{2,5} - \beta_{1,4} &\geq 0, \end{aligned}$$

which can be viewed as the following functionals being nonnegative on those Betti tables generated by our rays:

$$\begin{pmatrix} - & - & - \\ - & - & - \\ - & -1 & 1 \\ - & -2 & 2 \end{pmatrix} \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & -1 \\ - & - & 1 \end{pmatrix} \begin{pmatrix} - & - & - \\ - & - & - \\ - & -1 & - \\ - & - & 1 \end{pmatrix}$$

The Herzog-Kühl equations in this case are equivalent to the following system:

$$\begin{aligned} \beta_{0,0} + \beta_{2,4} &= \beta_{1,2} + \beta_{1,4} \\ \beta_{2,3} + \beta_{2,5} &= \beta_{1,1} + \beta_{1,3} \\ 2\beta_{0,0} + \beta_{2,3} &= 2\beta_{1,1} + \beta_{1,2} + \beta_{1,3} \end{aligned}$$

To give the proof that the inequalities above hold for all Betti tables, let M be a graded R-module whose Betti table is of the above shape. The Herzog-Kühl equations show that the first inequality is equivalent to the inequality $\beta_{1,4} + \beta_{2,3} \leq \beta_{0,0}$. If $\beta_{2,3}$ is nonzero, then the complex is a direct sum of $\beta_{2,3}$ copies of the Koszul complex on x and y, and another complex, and so we may assume that $\beta_{2,3}$ is zero by subtracting as many copies of the first ray from the Betti table as we can. Also, we know that the map from $R(-4)^{\beta_{1,4}}$ to $R^{\beta_{0,0}}$ is injective. Indeed, any element of the kernel would have degree greater than or equal to 6, since there are no linear syzygies among forms of degree 4 in R. Therefore, one has the desired inequality.

Also, note that $\beta_{2,5} \geq \beta_{1,3}$ is equivalent to $\beta_{2,3} \leq \beta_{1,1}$ from the second Herzog-Kühl equation above. Again, since the only way to get contributions to $\beta_{2,3}$ is from the Koszul complex, one has $\beta_{2,3} \leq \beta_{1,1}$.

To see the final inequality, note that one has $\beta_{2,5} \geq \beta_{1,4}$ since the dual of a Betti table is again a Betti table, and one may not have more linear syzygies than generators in degree 0, since there is only one variable of degree 1 in R.

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