Affine Toric Equivalence Relations are Effective

Claudiu Raicu
Department of Mathematics, University of California, Berkeley

Introduction

Any map of schemes $X \to Y$ defines an equivalence relation $R = X \times_Y X \to X \times X$, the relation of "being in the same fiber". János Kollár has asked whether all finite equivalence relations have this form, namely: given an S-scheme X and a finite scheme theoretic equivalence relation $R \subset X \times_S X$, does there exist an S-scheme Y and a finite surjective map $X \to Y$ over S such that $R \simeq X \times_Y X$? The answer to this question is in general negative, but is affirmative in the case of toric equivalence relations on affine toric varieties. One example of an equivalence relation not given by a map comes (as always) from Hironaka's construction of a proper nonprojective scheme ([Har77], p.443), but there are other easier and very explicit examples coming from the nonvanishing of the first Amitsur cohomology group associated to certain maps of algebras.

Equivalence Relations

Given a scheme X over a base S, a *scheme theoretic equivalence relation* on X over S is an S-scheme R together with a morphism $f : R \to X \times_S X$ over S such that for any S-scheme T, the set map $f(T) : R(T) \to X(T) \times X(T)$ is injective and its image is the graph of an equivalence relation on $X(T)$ (where for S-schemes Z, T we denote by $Z(T)$ the set of S-maps from T to Z).

In the affine case, we get a more down to earth description of equivalence relations. If k is a field and $X = \mathbb{A}^n_k$ is the n-dimensional affine space over k, then $\mathcal{O}_X \simeq k[\boldsymbol{x}]$, where $\boldsymbol{x} = (x_1, \cdots, x_n)$. To give an equivalence relation $R \subset X \times_k X$ is the same as giving an ideal $I(\boldsymbol{x}, \boldsymbol{y}) \subset k[\boldsymbol{x}, \boldsymbol{y}]$ that satisfies the following properties:

1. (reflexivity) $I(\boldsymbol{x}, \boldsymbol{y}) \subset (x_1 - y_1, \cdots, x_n - y_n)$;
2. (symmetry) $I(\boldsymbol{x}, \boldsymbol{y}) = I(\boldsymbol{y}, \boldsymbol{x})$;
3. (transitivity) $I(\boldsymbol{x}, \boldsymbol{z}) \subset I(\boldsymbol{x}, \boldsymbol{y}) + I(\boldsymbol{y}, \boldsymbol{z})$;

R is finite if and only if I satisfies

4. (finiteness) $k[\boldsymbol{x}, \boldsymbol{y}]/I(\boldsymbol{x}, \boldsymbol{y})$ is finite over $k[\boldsymbol{x}]$.

Any map $X \to Y$ defines an equivalence relation $R = X \times_Y X$, which we call *effective*. In the affine case effectivity corresponds to the ideal $I(\boldsymbol{x}, \boldsymbol{y})$ being generated by differences $f(\boldsymbol{x}) - f(\boldsymbol{y})$.

Toric Equivalence Relations

If X is a (not necessarily normal) toric variety, an equivalence relation R on X is said to be *toric* if it is invariant under the diagonal action of the torus. In the affine case, this suffices to insure effectivity:

> ***Theorem A.*** *Let k be a field, X/k an affine toric variety, and R a toric equivalence relation on X. Then there exists an affine toric variety Y together with a toric map $X \to Y$ such that $R \simeq X \times_Y X$.*

However, in the nonaffine case this is no longer true: an equivalence relation on $X = \mathbb{P}^2$ identifying the points of a (torus-invariant) line L can't be effective; if it were, then the map $X \to Y$ defining it would have to contract L and therefore be constant. Kollár proved that quotients by finite equivalence relations exist in positive characteristic ([Kol08]). This is not known in general, but is true in the case of finite affine toric equivalence relations ([Rai09], Cor. 5.3).

The Amitsur Complex

Given a commutative ring A and an A-algebra B, we consider the *Amitsur complex*

$$C(A, B): \ B \to B \otimes_A B \to \cdots \to B^{\otimes_A m} \to \cdots$$

starting in degree zero, with differentials given by the formula

$$d_{m-1}(b_1 \otimes b_2 \otimes \cdots \otimes b_m) = \sum_{i=1}^{m+1} (-1)^i b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_m.$$

It is well known that if B is a faithfully flat or augmented A-algebra, then $C(A, B)$ is exact (see [FD], Lemma 2.6). By *Theorem B* below this is also true when A, B are monoid rings (and the map $A \to B$ is defined on the monoid level).

> ***Theorem B.*** *Let k be any commutative ring, let τ and σ be commutative monoids, and let $\varphi : \tau \to \sigma$ be a map of monoids. If $A = k[\tau]$, $B = k[\sigma]$, and B is considered as an A-algebra via the map $A \to B$ induced by φ, then the Amitsur complex $C(A, B)$ is exact.*

Zig-zags

The kernel A' of the zeroth differential in $C(A, B)$ is (in nice situations) the coordinate ring of the quotient of the equivalence relation induced by the map $\varphi : A \to B$. In general A' is larger than A, but they are equal when φ is flat or augmented. When $A = k[\tau]$ and $B = k[\sigma]$ are monoid rings, and φ is induced by a map $\tau \to \sigma$, $A' = k[\tau']$ is also a monoid ring, where $\tau' \subset \sigma$ can be described as follows (Isbell's Zig-zag theorem, [How95], Thm. 8.3.4): an element $s \in \sigma$ belongs to τ' if and only if there exist $a_1, \cdots, a_{n-1}, b_1, \cdots, b_{n-1} \in \sigma$ and $s_1, \cdots, s_n, t_1, \cdots, t_{n-1} \in \tau$ giving a "zig-zag"

$$a = a_1 s_1, \ s_1 = t_1 b_1,$$
$$a_i t_i = a_{i+1} s_{i+1}, \ s_{i+1} b_i = t_{i+1} b_{i+1} \ (1 \le i \le n-2),$$
$$a_{n-1} t_{n-1} = s_n, \ s_n b_{n-1} = a.$$

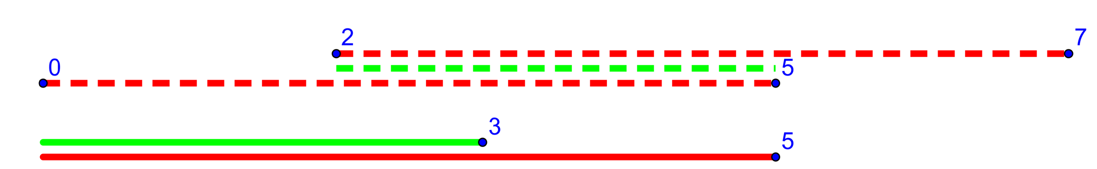

Figure 1: A 1-dimensional zig-zag

A minimal nontrivial zig-zag has $n = 2$. It turns out that when $\sigma = \mathbb{N}$ every element of τ' can be given by such a zig-zag. For example, if $\tau = \langle 3, 5 \rangle \subset \sigma = \mathbb{N}$, we get that $7 \in \tau' \setminus \tau$ via the zig-zag pictured in Figure 1. The submonoids $\tau \subset \mathbb{N}$ for which $\tau = \tau'$ are called *Arf semigroups*, and their associated monoid rings are particular examples of Arf rings. An *Arf ring* is essentially a 1-dimensional ring which is the quotient of the equivalence relation defined by the map to its normalization. For a beautiful story relating Arf rings to multiplicity sequences of curve branches see [Arf49] and [Lip71].

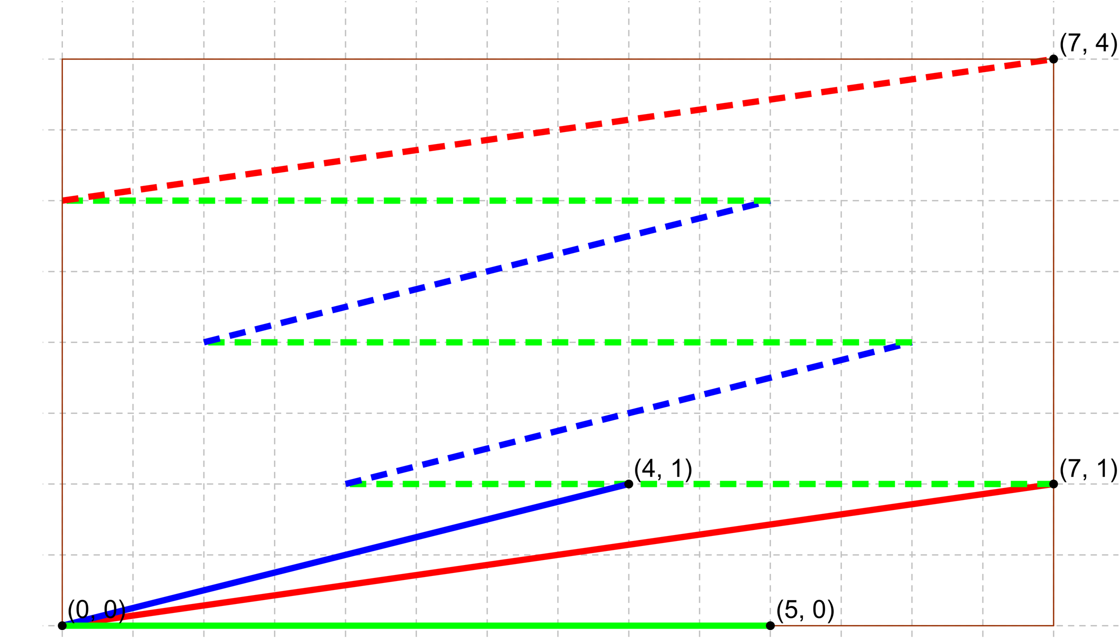

Figure 2: A 2-dimensional zig-zag

In contrast with the one-dimensional case, in higher dimension arbitrarily long zig-zags may be needed to recover the elements of τ'. For example, if we start with

$$\tau = \langle (n, 1), (n+1, 0), (2n-1, 1) \rangle \subset \sigma = \mathbb{N}^2,$$

we can see that $p = (2n-1, n) \in \tau' \setminus \tau$ and the minimal zig-zag producing p has length $2n - 1$. See Figure 2 for the case $n = 4$.

A Noneffective Equivalence Relation

If k is any ring, $A = k[f_1, \cdots, f_m] \subset B = k[\boldsymbol{x}]$, and $f(\boldsymbol{x}, \boldsymbol{y})$ is a 1-cocycle in $C(A, B)$, then the ideal

$$I(\boldsymbol{x}, \boldsymbol{y}) = (f(\boldsymbol{x}, \boldsymbol{y}), f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}), i = 1, \cdots, m)$$

defines an equivalence relation on Spec(B). When the f_i's are homogeneous, noneffectivity of this equivalence relation amounts to f not being a coboundary. If the f_i's are such that B is finite over A, then the equivalence relation is also finite. The following choice of f_i's yields an "universal" (i.e. characteristic free) noneffective equivalence relation:

$$f_1(\boldsymbol{x}) = \boldsymbol{x}_1^2, \ f_2(\boldsymbol{x}) = \boldsymbol{x}_1 \boldsymbol{x}_2 - \boldsymbol{x}_2^2, \ f_3(\boldsymbol{x}) = \boldsymbol{x}_2^3,$$
$$f(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}_1 \boldsymbol{y}_2 - \boldsymbol{x}_2 \boldsymbol{y}_1) \boldsymbol{y}_2^3.$$

References

[Arf49] C. Arf, *Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique*, Proc. London Math. Soc., Series 2, 50:256-287, 1949.

[FD] *Algebraic stacks project, Flat descent* http://www.math.columbia.edu/~dejong/algebraic_geometry/stacks-git/flat.pdf

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. Springer Verlag, New York, 1977.

[How95] J.M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.

[Kol08] J. Kollár, *Quotients by finite equivalence relations*, preprint (arXiv: 0812.3608).

[Lip71] J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. 93:649-685, 1971.

[Rai09] C. Raicu, *Affine toric equivalence relations are effective*, preprint (arXiv: 0905.4805).