The GSS Conjecture

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Introduction

The projectivization of the space of matrices of rank one coincides with the image of the Segre embedding of a product of two projective spaces. Its variety of secant $(r-1)$–planes is the space of matrices of rank at most r, whose equations are given by the $(r+1)\times(r+1)$ minors of a generic matrix. A fundamental problem, with applications in complexity theory and algebraic statistics, is to understand rank varieties of higher order tensors. This is a very complicated problem in general, even for the relatively small case of $\sigma_4(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3)$, the variety of secant 3–planes to the Segre product of three projective 3–spaces (also known as the Salmon Problem). Inspired by experiments related to Bayesian networks, Garcia, Stillman and Sturmfels ([GSS05]) gave a conjectural description of the ideal of the variety of secant lines to a Segre product of projective spaces. The case of an n–factor Segre product has been obtained for $n\le 5$ in a series of papers ([LM04],[LW07],[AR08]). We have proved the general case of the conjecture in ([Rai10]).

Flattenings and the GSS Conjecture

For a field K of characteristic zero and K–vector spaces $V_1,\cdots,V_n$, we consider the *Segre embedding*

$$\mathrm{Seg}:\mathbb{P}V_1^*\times\cdots\times\mathbb{P}V_n^*\to\mathbb{P}(V_1^*\otimes\cdots\otimes V_n^*),$$

given by $([e^1],\cdots,[e^n])\mapsto[e^1\otimes\cdots\otimes e^n]$. We write X for the image of this map. Its *k–th secant variety* is the closure of the set

$$\left\{\left[\sum_{i=0}^k c_i\cdot v_i^1\otimes\cdots\otimes v_i^n\right]: c_i\in K,\ v_i^j\in V_j^*\right\}$$

in $\mathbb{P}(V_1^*\otimes\cdots\otimes V_n^*)$, which we denote by $\sigma_{k+1}(X)$. For a partition $I\sqcup J=\{1,\cdots,n\}=:[n]$, we write $V_I=\otimes_{i\in I}V_i$, $V_J=\otimes_{j\in J}V_j$, so that any tensor in $V_1\otimes\cdots\otimes V_n$ can be *flattened* to a 2–tensor, i.e. a matrix, in $V_I\otimes V_J$. We get

$$\sigma_k(X)\subset\sigma_k(\mathrm{Seg}(\mathbb{P}V_I^*\times\mathbb{P}V_J^*)),$$

so the $(k+1)\times(k+1)$ minors of the generic matrix in $V_I\otimes V_J$ give some of the equations for $\sigma_k(X)$.

GSS Conjecture. *The ideal of the variety of secant lines to X is generated by 3×3 minors of flattenings.*

Equations for $\sigma_k(X)$

The coordinate ring of $\mathbb{P}(V_1^*\otimes\cdots\otimes V_n^*)$ is the polynomial ring $R:=\mathrm{Sym}(V_1\otimes\cdots\otimes V_n)=K[z_\alpha]$, where $z_\alpha=x_{a_1,1}\otimes\cdots\otimes x_{a_n,n}$ for some bases $\mathcal{B}_j=\{x_{ij}\}_i$ of V_j, $\alpha=(a_1,\cdots,a_n)\in\mathbb{Z}^n_{\ge 0}$. The ideal of $\sigma_k(X)$ is a $GL(V):=GL(V_1)\times\cdots\times GL(V_n)$–subrepresentation of R. The irreducible $GL(V)$-representations that will concern us will have the form $S_\lambda V:=S_{\lambda^1}V_1\otimes\cdots\otimes S_{\lambda^n}V_n$, for λ^i partitions of some d, where S_{λ^i} are Schur functors. We write $\lambda\vdash^n d$.

Theorem A. *The GSS conjecture holds and*

$$K[\sigma_2(X)]_d=\bigoplus_{\lambda\vdash^n d}(S_\lambda V)^{m_\lambda}\ \textit{for}\ d\ge 0,$$

where m_λ is as follows. Consider

$$b=\max\{\lambda_2^1,\cdots,\lambda_2^n\}\text{ and }e=\lambda_2^1+\cdots+\lambda_2^n.$$

If some λ^i has more than two parts, or $e<2b$, then $m_\lambda=0$. If $e\ge d-1$, then $m_\lambda=\lfloor d/2\rfloor-b+1$, unless e is odd and d is even, in which case $m_\lambda=\lfloor d/2\rfloor-b$. If $e<d-1$ and $e\ge 2b$, then $m_\lambda=\lfloor(e+1)/2\rfloor-b+1$, unless e is odd, in which case $m_\lambda=\lfloor(e+1)/2\rfloor-b$.

For a partition $\mu=(\mu_1,\cdots,\mu_t)$ of $d>0$, written $\mu\vdash d$, we consider the set $\mathcal{P}_\mu$ of all partitions A of $[d]$ of shape μ, i.e. $A=\{A_1,\cdots,A_t\}$ with $|A_i|=\mu_i$ and $\sqcup_{i=1}^t A_i=[d]$. We write $\mu=(\mu_1^{i_1}\cdots\mu_s^{i_s})$ with $\mu_i\neq\mu_j$, and consider the map

$$\pi_\mu:R_d\longrightarrow\bigotimes_{j=1}^s S_{(i_j)}(S_{(\mu_j)}V_1\otimes\cdots\otimes S_{(\mu_j)}V_n),$$

given by

$$z_1\cdots z_d\mapsto\sum_{A\in\mathcal{P}_\mu}\bigotimes_{j=1}^s\prod_{\substack{B\in A\\|B|=\mu_j}}m(z_i:i\in B),$$

where m denotes the usual multiplication map

$$m:(V_1\otimes\cdots\otimes V_n)^{\otimes\mu_j}\longrightarrow S_{(\mu_j)}V_1\otimes\cdots\otimes S_{(\mu_j)}V_n.$$

The intersection of the kernels of the maps π_μ, for $\mu\vdash d$ with at most k parts, gives all the equations of degree d of $\sigma_k(X)$ ([Rai10], Prop. 3.3).

If $n=3$, $\mu=(2,1)$, $z_1=z_{(1,1,1)}$, $z_2=z_{(1,2,1)}$ and $z_3=z_{(2,1,2)}$, then

$$\pi_\mu(z_1\cdot z_2\cdot z_3)=z_{(\{1,1\},\{1,2\},\{1,1\})}\cdot z_{(2,1,2)}+$$
$$z_{(\{1,2\},\{1,1\},\{1,2\})}\cdot z_{(1,2,1)}+z_{(\{1,2\},\{1,2\},\{1,2\})}\cdot z_{(1,1,1)},$$

where $z_{(\{1,1\},\{1,2\},\{1,1\})}=x_{11}^2\otimes x_{12}x_{22}\otimes x_{13}^2$ etc.

The "Generic" Case

The induced representation $U_d^n:=\mathrm{Ind}_{S_d}^{S_d^n}(\mathbf{1})$, with $\mathbf{1}$ the trivial representation of the symmetric group S_d, and S_d included in S_d^n diagonally, has a basis of monomials $m=z_{(a_{11},\cdots,a_{1n})}\cdots z_{(a_{d1},\cdots,a_{dn})}$, with $\{a_{1j},\cdots,a_{dj}\}=[d]$ for each j. S_d^n acts on U_d^n by letting the j–th copy of S_d act on $\{a_{1j},\cdots,a_{dj}\}$. For fixed λ, we identify m with an *n–tableau* $T=T^1\otimes\cdots\otimes T^n$ of shape λ, where the a_{ij}–th box of T^j has entry i. We identify two n-tableaux if their entries differ by a permutation of $[d]$. E.g.

$$z_{(1,1,2)}\cdot z_{(2,3,1)}\cdot z_{(3,2,3)}=\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}$$

$$z_{(3,2,3)}\cdot z_{(1,1,2)}\cdot z_{(2,3,1)}=\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}$$

For $\mu=(a\ge b)\vdash d$, we consider the space U_μ^n spanned by monomials $z_{(A_1,\cdots,A_n)}\cdot z_{(B_1,\cdots,B_n)}$, where $A_i\sqcup B_i=[d]$, with $|A_i|=a$, $|B_i|=b$. We define the map $\pi'_\mu:U_d^n\longrightarrow U_\mu^n$ by

$$\pi'_\mu(m)=\prod_{\substack{\{A,B\}\in\mathcal{P}_\mu,|A|=a,|B|=b\\A_j=\{a_{ij}:i\in A\},B_j=\{a_{ij}:i\in B\}}}z_{(A_1,\cdots,A_n)}\cdot z_{(B_1,\cdots,B_n)},$$

For $m=z_{(1,1,2)}\cdot z_{(2,3,1)}\cdot z_{(3,2,3)}$ and $\mu=(2,1)$

$$\pi'_\mu(m)=z_{(\{1,2\},\{1,3\},\{1,2\})}\cdot z_{(3,2,3)}+$$
$$z_{(\{1,3\},\{1,2\},\{2,3\})}\cdot z_{(2,3,1)}+z_{(\{2,3\},\{2,3\},\{1,3\})}\cdot z_{(1,1,2)}.$$

For $I\sqcup J=[n]$, $F_{I,J}\subset U_d^n$ denotes the "ideal" of 3×3 minors of *generic (I,J)–flattenings*

$$F_{I,J}=\mathrm{Span}([\alpha_1,\alpha_2,\alpha_3|\beta_1,\beta_2,\beta_3]\cdot z_{\gamma_4}\cdots z_{\gamma_d}),$$

where $[\alpha_1,\alpha_2,\alpha_3|\beta_1,\beta_2,\beta_3]=\det(z_{\alpha_i+\beta_j})$, $\alpha_i=(a_{ij})_{j=1}^n$, $\beta_i=(b_{ij})_{j=1}^n$, $\gamma_i=(c_{ij})_{j=1}^n$, with $a_{ij}=0$ for $i\in J$ and $b_{ij}=0$ for $i\in I$, and for each j, $\cup_i\{a_{ij},b_{ij},c_{ij}\}=\{0,1,\cdots,d\}$. For $n=d=3$,

$$\begin{array}{c}[(1,1,0),(2,3,0),\\(3,2,0)|(0,0,2),\\(0,0,1),(0,0,3)]\end{array}=\begin{vmatrix}z_{(1,1,2)}&z_{(1,1,1)}&z_{(1,1,3)}\\z_{(2,3,2)}&z_{(2,3,1)}&z_{(2,3,3)}\\z_{(3,2,2)}&z_{(3,2,1)}&z_{(3,2,3)}\end{vmatrix}.$$

We let $F_d^n:=\Sigma F_{I,J}$ be the space of all generic flattenings and $I_d^n:=\cap_\mu\mathrm{Ker}(\pi'_\mu)$ for $\mu=(a\ge b)\vdash d$.

Generic GSS

Theorem A follows by a specialization argument from its generic analogue below (V_λ denotes the irreducible S_d^n–representation corresponding to $\lambda\vdash^n d$)

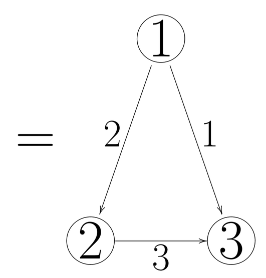
Theorem B. *$F_d^n=I_d^n$ and*

$$U_d^n/I_d^n=\bigoplus_{\lambda\vdash^n d}V_\lambda^{m_\lambda}\ \textit{for}\ d\ge 0,$$

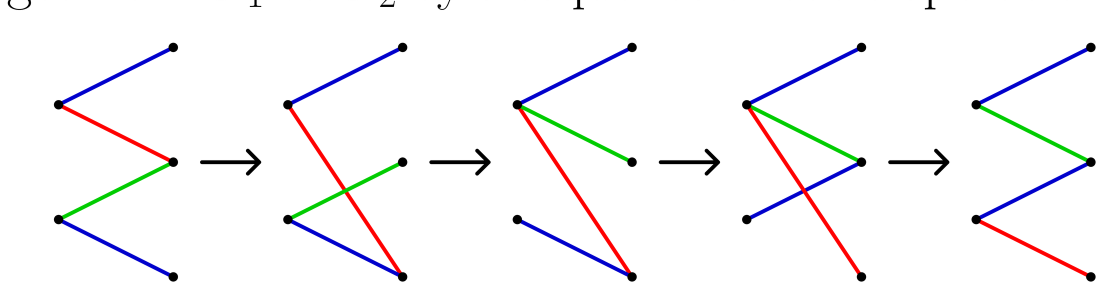
where m_λ is as in Theorem A.

Let c_λ be the Young symmetrizer corresponding to λ. If some $\lambda_3^i>0$, $Q_\lambda:=c_\lambda\cdot(U_d^n/F_d^n)=0$. Otherwise, we associate to an n–tableau T of shape λ a graph G: for each column $\begin{array}{|c|}\hline x\\\hline y\\\hline\end{array}$ of some T^i, G has an edge (x,y) of color i. If we write $G=\widehat{c_\lambda\cdot T}\in Q_\lambda$,

$$c_\lambda\cdot\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}=$$



$G=0$ if it contains an odd cycle, or if it has a bipartition (a,a), an odd number of edges, and is connected. $G_1=\pm G_2$ if they are connected and bipartite, with the same bipartition $(a\ge b)$. We get from G_1 to G_2 by a sequence of basic operations:



A set $\mathcal{B}$ of nonzero G's, which are connected and bipartite with distinct bipartitions, generates Q_λ and its image via $\oplus_\mu\pi'_\mu$ is a linearly independent set, hence $F_d^n=I_d^n$, and $\mathcal{B}$ is a basis of Q_λ.

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