Affine Toric Equivalence Relations are Effective

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Motivating Question

Under what circumstances do quotients by finite equivalence relations exist?

Outline of talk:

1. Equivalence Relations
2. The Amitsur Complex
3. A Noneffective Equivalence Relation
4. Questions
Definition of Equivalence Relations

Given a scheme $X$ over a base $S$, a scheme theoretic equivalence relation on $X$ over $S$ is an $S$-scheme $R$ together with a morphism

$$f : R \to X \times_S X$$

over $S$ such that for any $S$-scheme $T$, the set map

$$f(T) : R(T) \to X(T) \times X(T)$$

is injective and its image is the graph of an equivalence relation on $X(T)$ (here $Z(T)$ denotes the set of $S$-maps from $T$ to $Z$).
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$$R \rightrightarrows X$$

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is injective and its image is the graph of an equivalence relation on $X(T)$ (here $Z(T)$ denotes the set of $S$-maps from $T$ to $Z$). $R$ is said to be **finite** if the two projections

$$R \to X$$

are finite. A coequalizer of this two projections is called the **quotient** of $X$ by the equivalence relation $R$. 
The Affine Case

If $k$ is a field and $X = \mathbb{A}^n_k$ is the $n$-dimensional affine space over $k$, then $\mathcal{O}_X \cong k[x]$, where $x = (x_1, \cdots, x_n)$. An equivalence relation $R \subset X \times_k X$ corresponds to an ideal $I(x, y) \subset k[x, y]$.
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1. \textit{(reflexivity)}
   \[ I(x, y) \subset (x_1 - y_1, \cdots, x_n - y_n) \]

2. \textit{(symmetry)}
   \[ I(x, y) = I(y, x) \]
The Affine Case

If \( k \) is a field and \( X = \mathbb{A}^n_k \) is the \( n \)-dimensional affine space over \( k \), then \( \mathcal{O}_X \cong k[x] \), where \( x = (x_1, \cdots, x_n) \). An equivalence relation \( R \subset X \times_k X \) corresponds to an ideal \( l(x, y) \subset k[x, y] \) satisfying:

1. (reflexivity) \[ l(x, y) \subset (x_1 - y_1, \cdots, x_n - y_n) \]

2. (symmetry) \[ l(x, y) = l(y, x) \]

3. (transitivity) \[ l(x, z) \subset l(x, y) + l(y, z) \]
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\( R \) is finite if and only if \( l \) satisfies

4. (finiteness) \[
k[x, y]/l(x, y) \text{ is finite over } k[x]
\]
Effective Equivalence Relations

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An equivalence relation $R$ on $X$ is said to be effective if there exists a morphism $X \rightarrow Y$ such that

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In the affine case effectivity corresponds to the ideal $I(x, y)$ of the equivalence relation being generated by differences $f(x) - f(y)$.
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“**Theorem**” If \( X, Y \) and \( f : X \rightarrow Y \) are “nice”, and if it happens that the effective equivalence relation \( R = X \times_Y X \) defined by \( f \) is finite, then the quotient \( X/R \) exists.
Toric Equivalence Relations

If $X$ is a (not necessarily normal) toric variety, an equivalence relation $R$ on $X$ is said to be **toric** if it is invariant under the diagonal action of the torus.

Remarks:
Theorem (–, 2009) Let $k$ be a field, $X/k$ an affine toric variety, and $R$ a toric equivalence relation on $X$. Then there exists an affine toric variety $Y$ together with a toric map $X \to Y$ such that $R \cong X \times Y$. 

If $R$ is finite, the quotient exists and is also an affine toric variety. The theorem is false in the nonaffine case: an equivalence relation on $X = \mathbb{P}^2$ identifying the points of a (torus-invariant) line $L$ can't be effective; if it were, then the map $X \to Y$ defining it would have to contract $L$ and therefore be constant.
Toric Equivalence Relations

If \( X \) is a (not necessarily normal) toric variety, an equivalence relation \( R \) on \( X \) is said to be \textbf{toric} if it is invariant under the diagonal action of the torus. In the affine case, this suffices to insure effectivity:

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Definition of the Amitsur Complex

Given a commutative ring $A$ and an $A$-algebra $B$, we consider the Amitsur complex

$$ C(A, B) : B \rightarrow B \otimes_A B \rightarrow \cdots \rightarrow B^\otimes_A m \rightarrow \cdots $$

with differentials given by the formula

$$ d(b_1 \otimes b_2 \otimes \cdots \otimes b_m) = \sum_{i=1}^{m+1} (-1)^i b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_m. $$
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It is well known that if $B$ is a faithfully flat or augmented $A$-algebra, then $C(A, B)$ is exact. In these cases, the kernel of the first differential is $A$. 
Exactness of the Amitsur Complex

It turns out that exactness holds also when $A, B$ are monoid rings and the map $A \to B$ is defined on the monoid level:

**Theorem (–, 2009)**

Let $k$ be any commutative ring, let $\tau$ and $\sigma$ be commutative monoids, and let $\varphi : \tau \to \sigma$ be a map of monoids. If $A = k[\tau], B = k[\sigma]$, and $B$ is considered as an $A$-algebra via the map $A \to B$ induced by $\varphi$, then the Amitsur complex $C(A, B)$ is exact.
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As opposed to the faithfully flat and augmented cases, the kernel of the first differential

$$d : B \to B \otimes_A B, \quad b \mapsto b \otimes 1 - 1 \otimes b$$

is usually larger than $A$. 
A 1–Dimensional Zig–zag

If we consider

\[ A = k[t^3, t^5] \subset B = k[t] \]

then \( t^7 \in B \) is not an element of \( A \), but it goes to zero under the first differential in the Amitsur complex.
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A 2–Dimensional Zig–zag
A Noneffective Affine Equivalence Relation

If $k$ is any ring, $A = k[f_1, \cdots, f_m] \subset B = k[x]$, and $f(x, y)$ is a 1–cocycle in the Amitsur complex $C(A, B)$, i.e.

$$f(y, z) - f(x, z) + f(x, y) = 0 \in k[x, y, z]/(f_i(x) - f_i(y), f_i(x) - f_i(z)),$$

then the ideal

$$I(x, y) = (f(x, y), f_i(x) - f_i(y) : i = 1, \cdots, m) \subset k[x, y]$$

defines an equivalence relation on $\text{Spec}(B)$. When the $f_i$’s are homogeneous, noneffectivity of this equivalence relation amounts to $f$ not being a coboundary.
A Noneffective Affine Equivalence Relation

If $k$ is any ring, $A = k[f_1, \cdots, f_m] \subset B = k[x]$, and $f(x, y)$ is a 1–cocycle in the Amitsur complex $C(A, B)$, i.e.

$$f(y, z) - f(x, z) + f(x, y) = 0 \in k[x, y, z]/(f_i(x) - f_i(y), f_i(x) - f_i(z)),$$

then the ideal

$$I(x, y) = (f(x, y), f_i(x) - f_i(y) : i = 1, \cdots, m) \subset k[x, y]$$

defines an equivalence relation on Spec($B$). When the $f_i$’s are homogeneous, noneffectivity of this equivalence relation amounts to $f$ not being a coboundary.

Example

$$f_1(x) = x_1^2, \ f_2(x) = x_1x_2 - x_2^2, \ f_3(x) = x_2^3,$$

$$f(x, y) = (x_1y_2 - x_2y_1)y_2^3.$$
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- Is there a geometric way of explaining the noneffective equivalence relations coming from the nonvanishing of the first cohomology of the Amitsur complex?