Local cohomology with support in determinantal ideals

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Resolutions

Example

$I_2 = 2 \times 2$ minors of a 3×3 matrix. $S = \mathrm{Sym}(\mathbb{C}^3 \otimes \mathbb{C}^3)$.

$$\beta(S/I_2): \qquad \begin{array}{ccccc} 1 & . & . & . & . \\ . & 9 & 16 & 9 & . \\ . & . & . & . & 1 \end{array}$$

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More generally, $I_p = p \times p$ minors of $m \times n$ matrix, $S = \mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)$.

$$\beta(S/I_p): \text{ Lascoux, Józefiak, Pragacz, Weyman '80.}$$

Feature: I_p is a $\mathrm{GL}_m \times \mathrm{GL}_n$–representation. Assume $m \geq n$.

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Cauchy's formula: $$S = \bigoplus_{\underline{x}=(x_1 \geq x_2 \geq \cdots \geq x_n)} S_{\underline{x}}\mathbb{C}^m \otimes S_{\underline{x}}\mathbb{C}^n.$$

$$I_p = \left(\bigwedge^p \mathbb{C}^m \otimes \bigwedge^p \mathbb{C}^n \right) = \left(S_{(1^p)}\mathbb{C}^m \otimes S_{(1^p)}\mathbb{C}^n \right).$$

The ideals $I_{\underline{x}}$

$$I_{\underline{x}} = (S_{\underline{x}}\mathbb{C}^m \otimes S_{\underline{x}}\mathbb{C}^n) = \bigoplus_{\underline{x} \subset \underline{y}} S_{\underline{y}}\mathbb{C}^m \otimes S_{\underline{y}}\mathbb{C}^n.$$

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Example

$m = n = 3$, $I_{\underline{x}} = I_{(2,2)}$.

$$\beta(S/I_{2,2}) : \quad \begin{array}{ccccccc}
1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & 36 & 90 & 84 & 36 & 9 & 1 \\
. & . & . & . & . & . & . \\
. & . & . & . & 1 & . & .
\end{array}$$

Regularity of the ideals $I_{\underline{x}}$

Unfortunately, we don't know how to compute $\beta(S/I_{\underline{x}})$ for arbitrary $\underline{x}$!

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Theorem (–, Weyman '13)

$$reg(I_{\underline{x}}) = \max_{\substack{p=1,\cdots,n \\ x_p > x_{p+1}}} (n \cdot (x_p - p) + p^2 + 2 \cdot (p-1) \cdot (n-p)).$$

In particular, the only ideals $I_{\underline{x}}$ which have a linear resolution are those for which $x_1 = \cdots = x_n$ or $x_1 - 1 = x_2 = \cdots = x_n$.

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Example

For $m = n = 3$,

$$\mathrm{reg}(I_{(1,1)}) = 3,$$

$$\mathrm{reg}(I_{(2,2)}) = 6.$$

Local cohomology

The Čech complex $\mathcal{C}^\bullet(f_1, \cdots, f_t)$ is defined by

$$0 \longrightarrow S \longrightarrow \bigoplus_{1 \le i \le t} S_{f_i} \longrightarrow \bigoplus_{1 \le i < j \le t} S_{f_i f_j} \longrightarrow \cdots \longrightarrow S_{f_1 \cdots f_t} \longrightarrow 0.$$

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For $I = (f_1, \cdots, f_t)$, $i \ge 0$, the local cohomology modules $H_I^i(S)$ are defined by

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Note that

$$H^\bullet_I(S) = H^\bullet_{\sqrt{I}}(S),$$

and

$$\sqrt{I_{\underline{x}}} = I_p,$$

where p is the number of non-zero parts of $\underline{x}$.

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$H^\bullet_{I_p}(S)$ is an example of a doubly-graded module M_i^j, equivariant with respect to the action of $\mathrm{GL}_m \times \mathrm{GL}_n$.

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For such M, we define the character χ_M by

$$\chi_M(z, w) = \sum_{i,j} [M_i^j] \cdot z^i \cdot w^j,$$

where $[M_i^j]$ is the class of M_i^j in the representation ring of $\mathrm{GL}_m \times \mathrm{GL}_n$.

Dominant weights

We define the set of dominant weights in $\mathbb{Z}^r$ (for $r = m$ or n)

$$\mathbb{Z}^r_{dom} = \{\lambda \in \mathbb{Z}^r : \lambda_1 \geq \cdots \geq \lambda_r\}.$$

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For $\lambda \in \mathbb{Z}^n_{dom}$, $s = 0, 1, \cdots, n-1$, let

$$\lambda(s) = (\lambda_1, \cdots, \lambda_s, \underbrace{s-n, \cdots, s-n}_{m-n}, \lambda_{s+1} + (m-n), \cdots, \lambda_n + (m-n)).$$

For $m > n$, $\lambda(s)$ is dominant if and only if $\lambda_s \geq s - n$ and $\lambda_{s+1} \leq s - m$.

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For $m > n$, $\lambda(s)$ is dominant if and only if $\lambda_s \geq s - n$ and $\lambda_{s+1} \leq s - m$. The following Laurent power series are the key players in the description of $H^\bullet_{l_p}(S)$.

$$h_s(z) = \sum_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_s \geq s-n \\ \lambda_{s+1} \leq s-m}} [S_{\lambda(s)}\mathbb{C}^m \otimes S_\lambda \mathbb{C}^n] \cdot z^{|\lambda|}.$$

An example

Take $m = 11$, $n = 9$, $s = 5$, $\lambda = (4, 2, 1, -2, -3, -6, -8, -8, -10)$. We have $m - n = 2$ and

$$\lambda(s) = (\lambda_1, \cdots, \lambda_s, \underbrace{s - n, \cdots, s - n}_{m-n}, \lambda_{s+1} + (m - n), \cdots, \lambda_n + (m - n))$$

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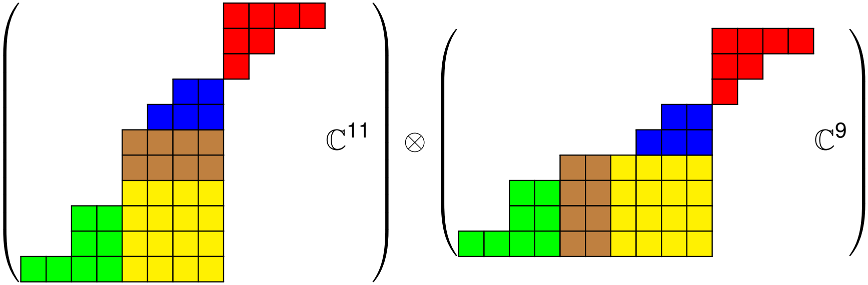
The coefficient of z^{-30} in $h_s(z)$ involves (among other terms)



Local cohomology with support in determinantal ideals

Theorem (–, Weyman, Witt '13)

$$\chi_{H^{\bullet}_{I_n}(S)}(z,w) = \sum_{s=0}^{n-1} h_s(z) \cdot w^{1+(n-s)\cdot(m-n)}.$$

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We define the Gauss polynomial $\binom{a}{b}$ to be the generating function

$$\binom{a}{b}(w) = \sum_{b\geq t_1\geq t_2\geq\cdots\geq t_{a-b}\geq 0} w^{t_1+\cdots+t_{a-b}} = \sum_{c\geq 0} p(a-b,b;c)\cdot w^c,$$

where $p(a-b,b;c) = \#\{\underline{t}\vdash c : \underline{t}\subset (b^{a-b})\}$.

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Theorem (–, Weyman '13)

$$\chi_{H^\bullet_{I_p}(S)}(z,w) = \sum_{s=0}^{p-1} h_s(z)\cdot w^{(n-p+1)^2+(n-s)\cdot(m-n)}\cdot\binom{n-s-1}{p-s-1}(w^2).$$

Ext modules

$$\operatorname{reg}(M) = \max\{-r - j : \operatorname{Ext}^j_S(M, S)_r \neq 0\}.$$

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Theorem (–, Weyman '13)

Consider partitions $\underline{x}, \underline{y}$, where $\underline{x}$ is obtained by removing some of the columns at the end of $\underline{y}$. The natural quotient map $S/I_{\underline{y}} \twoheadrightarrow S/I_{\underline{x}}$ induces injective maps

$$\mathrm{Ext}^i_S(S/I_{\underline{x}}, S) \hookrightarrow \mathrm{Ext}^i_S(S/I_{\underline{y}}, S) \quad \forall i.$$