# Local Cohomology with Support in Ideals of Maximal Minors and sub–Maximal Pfaffians

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# Overview

1. Cohen–Macaulayness of modules of covariants
2. Local cohomology
3. Ext modules via the geometric technique and duality

# Modules of covariants

## Theorem (Hochster–Roberts ’74)

*Consider a reductive group $H$ in characteristic zero, and a finite dimensional $H$–representation $W$. Write $S = \mathrm{Sym}(W)$, and let $S^H$ be the ring of invariants with respect to the natural action of $H$ on $S$. $S^H$ is a Cohen–Macaulay ring.*

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*Which modules of covariants are Cohen–Macaulay?*

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*Which modules of covariants are Cohen–Macaulay?*

[Stanley '82, Brion '93, Van den Bergh '90s.] For us:

- $G$ finite dimensional vector space, $\dim(G) = n$.
- $H = \text{SL}(G)$.
- $W = G^{\oplus m}$.

# Theorem on covariants of the special linear group

$S = \mathrm{Sym}(W) = \mathbb{C}[x_{ij}]$, where $x_{ij}$ are the entries of the generic matrix

$$X = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{mn} \end{bmatrix}.$$

$$S^H = \mathbb{C}[n \times n \text{ minors of } X] = \begin{cases} \mathbb{C}, & m < n; \\ \mathbb{C}[\det(X)], & m = n; \\ \text{more interesting}, & m > n. \end{cases}$$

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**Theorem (–WW '13)**

*If $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0)$ is a partition and $U = S_\mu G$, then $(S \otimes U)^H$ is Cohen–Macaulay if and only if $\mu_s - \mu_{s+1} < m - n$ for all $s = 1, \cdots, n-1$.*

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[B'93: $m = n + 1$; VdB'94: $n = 2$, arbitrary $W$; VdB'99: $n = 3$.]

# Local cohomology

If $R$ is a ring, $J = (f_1, \cdots, f_t)$ an ideal, and $M$ an $R$–module, we define the Čech complex $\mathcal{C}^\bullet(M; f_1, \cdots, f_t)$ by

$$0 \longrightarrow M \longrightarrow \bigoplus_{1\le i\le t} M_{f_i} \longrightarrow \bigoplus_{1\le i<j\le t} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1\cdots f_t} \longrightarrow 0.$$

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For $j \ge 0$, the local cohomology modules $H_J^j(M)$ are defined by

$$H_J^j(M) = H^j(\mathcal{C}^\bullet(M; f_1, \cdots, f_t)).$$

If $R$ is local or graded, with maximal ideal $\mathfrak{m}$, then $M$ is said to be Cohen–Macaulay if

$$H_{\mathfrak{m}}^j(M) = 0 \text{ for } j < \dim(M).$$

For us $t = \binom{m}{n}$, $f_1, \cdots, f_t$ are the maximal minors of $X$, $R = S^H$, $\mathfrak{m} = (f_1, \cdots, f_t) \subset R$ is the homogeneous maximal ideal. We have

$$H_{\mathfrak{m}}^j\left((S \otimes U)^H\right) = \left(H_{\mathfrak{m}S}^j(S) \otimes U\right)^H.$$

# Local cohomology and covariants

Recall that $H = \mathrm{SL}(G)$, $W = G^{\oplus m}$, $S = \mathrm{Sym}(W)$, and $X$ is the generic $m \times n$ matrix. We have

$$S^H = \mathbb{C}[\text{maximal minors of } X] = \mathbb{C}[\text{Grass}(n, m)],$$

so for every $H$–representation $U$,

$$\dim(S^H) = \dim(S \otimes U)^H = n \cdot (m - n) + 1.$$

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Let $I \subset S$ be the ideal generated by the maximal minors of $X$. It follows that $(S \otimes U)^H$ is Cohen–Macaulay iff

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When $U = S_\mu G$ is an irreducible $H$–representation, this is equivalent to saying that $U^* = S_{(\mu_1, \mu_1 - \mu_{n-1}, \cdots, \mu_1 - \mu_2)} G$ doesn't occur in the decomposition of $H_I^j(S)$ into a sum of irreducible $H$–representations.

# Theorem on Maximal Minors

Write $G^{\oplus m} = F \otimes G$ for an $m$–dimensional vector space $F$, so that $S = \mathrm{Sym}(F \otimes G)$. $I$ is generated by $\bigwedge^n F \otimes \bigwedge^n G \subset \mathrm{Sym}^n(F \otimes G)$.

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**Theorem (–WW '13)**

*For $1 \leq s \leq n$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ a dominant weight, let*

$$\lambda(s) = (\lambda_1, \cdots, \lambda_{n-s}, \underbrace{-s, \cdots, -s}_{m-n}, \lambda_{n-s+1} + (m-n), \cdots, \lambda_n + (m-n)).$$

*We let $W(r; s)$ denote the set of dominant weights $\lambda \in \mathbb{Z}^n$ with $|\lambda| = r$ and $\lambda(s) \in \mathbb{Z}^m$ also dominant. We have the decomposition into a sum of $\mathrm{GL}(F) \times \mathrm{GL}(G)$–representations*

$$H_I^j(S)_r = \begin{cases} \bigoplus_{\lambda \in W(r;s)} S_{\lambda(s)} F \otimes S_\lambda G, & \text{if } j = s \cdot (m-n) + 1, 1 \leq s \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

# Weights of local cohomology for maximal minors

Take $m = 11$, $n = 9$, $s = 4$, $\lambda = (4, 2, 1, -2, -3, -6, -8, -8, -10)$. We have $m - n = 2$ and

$$\lambda(s) = (\lambda_1, \cdots, \lambda_{n-s}, \underbrace{-s, \cdots, -s}_{m-n}, \lambda_{n-s+1} + (m-n), \cdots, \lambda_n + (m-n))$$

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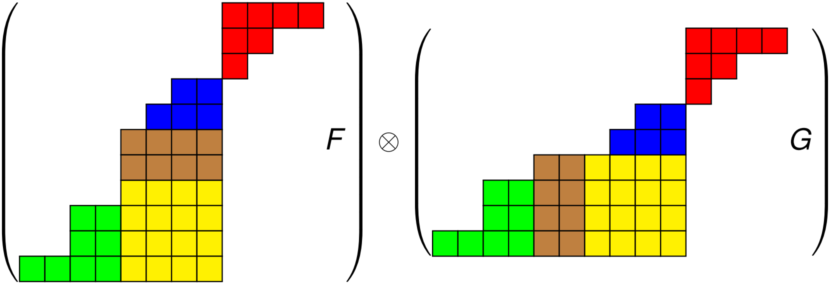
$$= (4, 2, 1, -2, -3, -4, -4, -4, -6, -6, -8).$$

The local cohomology module $H_I^9(S)$ contains in degree $r = |\lambda| = -30$ the irreducible representation



# Theorem on sub–Maximal Pfaffians

$\dim(F) = 2n+1$, $W = \bigwedge^2 F$, and $S = \mathrm{Sym}(W)$. Let $I$ be the ideal generated by $\bigwedge^{2n} F \subset \mathrm{Sym}^n\left(\bigwedge^2 F\right)$ (the $2n \times 2n$–Pfaffians of the generic $(2n+1) \times (2n+1)$ skew–symmetric matrix).

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$$\lambda(s) = (\lambda_1, \cdots, \lambda_{2n-2s}, -2s, \lambda_{2n-2s+1} + 1, \cdots, \lambda_{2n} + 1).$$

*We let $W(r;s)$ denote the set of dominant weights $\lambda \in \mathbb{Z}^{2n}$ with $|\lambda| = 2r$, satisfying $\lambda_{2i-1} = \lambda_{2i}$ for $i = 1, \cdots, n$, and such that $\lambda(s) \in \mathbb{Z}^{2n+1}$ is also dominant. We have the decomposition into a sum of $\mathrm{GL}(F)$–representations*

$$H_I^j(S) = \begin{cases} \bigoplus_{\lambda \in W(r;s)} S_{\lambda(s)} F, & \text{if } j = 2s+1, 1 \leq s \leq n; \\ 0, & otherwise. \end{cases}$$

# Weights of local cohomology for Pfaffians

Take $n = 5$, $s = 2$, $\lambda = (5, 5, 2, 2, -3, -3, -6, -6, -9, -9)$. We have

$$\lambda(s) = (\lambda_1, \cdots, \lambda_{2n-2s}, -2s, \lambda_{2n-2s+1} + 1, \cdots, \lambda_{2n} + 1)$$

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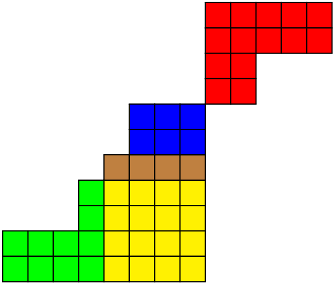
$$= (5, 5, 2, 2, -3, -3, -4, -5, -5, -8, -8).$$

The local cohomology module $H^5_I(S)$ contains in degree $r = |\lambda| = -22$ the irreducible representation



$F$

# Local cohomology and Ext modules

The local cohomology modules $H_I^j(S)$ can be computed via

$$H_I^j(S) = \varinjlim_{d} \mathrm{Ext}_S^j(S/I^d, S).$$

Moreover, we have $\mathrm{Ext}_S^j(S/I^d, S) = \mathrm{Ext}_S^{j-1}(I^d, S)$ for $j > 0$.

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One can realize $I^d$ as the global sections of a vector bundle with vanishing higher cohomology on a certain Grassmann variety, and then use duality to compute the relevant Ext modules.

More generally, consider a projective variety $X$, a finite dimensional vector space $W$, and an exact sequence

$$0 \longrightarrow \xi \longrightarrow W \otimes \mathcal{O}_X \longrightarrow \eta \longrightarrow 0,$$

where $\xi$ and $\eta$ are vector bundles on $X$.

# Theorem on Ext modules

For a vector bundle $\mathcal{V}$ on $X$, define

$$\mathcal{M}(\mathcal{V}) = \mathcal{V} \otimes \text{Sym}(\eta),$$

and

$$\mathcal{M}^*(\mathcal{V}) = \mathcal{V} \otimes \det(\xi) \otimes \text{Sym}(\eta^*).$$

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## Theorem (–WW '13)

*Assume that $H^j(X, \mathcal{M}(\mathcal{V})) = 0$ for $j > 0$, and let*

$$M(\mathcal{V}) = H^0(X, \mathcal{M}(\mathcal{V})).$$

*We have for $j \geq 0$ a graded isomorphism*

$$\text{Ext}^j_S(M(\mathcal{V}), S) = H^{rank(\xi)-j}(X, \mathcal{M}^*(\mathcal{V}))^*,$$

*where $(-)^*$ stands for the graded dual.*