Secant Varieties of Segre–Veronese Varieties

Claudiu Raicu

Princeton University

Raleigh, October 2011
Overview

1. Secant Varieties
2. Segre–Veronese Varieties
3. Flattenings
4. Main Results and Techniques
Secant Varieties

Definition

Given a subvariety $X \subset \mathbb{P}^N$, the $(k - 1)$–st secant variety of $X$, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by $k$ points on $X$:

$$\sigma_k(X) = \bigcup_{x_1, \ldots, x_k \in X} \mathbb{P}_{x_1, \ldots, x_k}.$$
Secant Varieties

Definition

Given a subvariety $X \subset \mathbb{P}^N$, the $(k - 1)$–st secant variety of $X$, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by $k$ points on $X$:

$$\sigma_k(X) = \bigcup_{x_1, \ldots, x_k \in X} \mathbb{P}^{x_1, \ldots, x_k}.$$

Alternatively, write $\mathbb{P}^N = \mathbb{P}W$ for some vector space $W$, and let $\hat{X} \subset W$ denote the cone over $X$. The cone $\sigma_k(X)$ over $\sigma_k(X)$ is the closure of the image of the map

$$s : \hat{X} \times \cdots \times \hat{X} \rightarrow W,$$

$$s(x_1, \cdots, x_k) = x_1 + \cdots + x_k.$$
Secant Varieties

Definition

Given a subvariety $X \subset \mathbb{P}^N$, the $(k - 1)$–st secant variety of $X$, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by $k$ points on $X$:

$$\sigma_k(X) = \bigcup_{x_1, \ldots, x_k \in X} \mathbb{P}^{x_1, \ldots , x_k}.$$

Alternatively, write $\mathbb{P}^N = \mathbb{P} W$ for some vector space $W$, and let $\hat{X} \subset W$ denote the cone over $X$. The cone $\sigma_k(X)$ over $\sigma_k(X)$ is the closure of the image of the map

$$s : \hat{X} \times \cdots \times \hat{X} \longrightarrow W,$$

$$s(x_1, \cdots , x_k) = x_1 + \cdots + x_k.$$

Problem

Given (the equations of) $X$, determine (the equations of) $\sigma_k(X)$. 
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}({\mathcal{W}^*}) \rightarrow K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$. 

Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}(W^*) \rightarrow K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*

Interesting examples:

1. curves;
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*

Interesting examples:

1. curves;
2. toric varieties;
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.  

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*

Interesting examples:

1. curves;
2. toric varieties;
3. homogeneous spaces;
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \operatorname{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*

Interesting examples:

1. curves;
2. toric varieties;
3. homogeneous spaces;
4. Grassmannians;
Solution to Problem

The morphism $s$ of affine varieties corresponds to a ring map

$$s^\# : \text{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^\#$.

Big Problem

*Computing the kernel and image of $s^\#$ is really hard!*

Interesting examples:

1. curves;
2. toric varieties;
3. homogeneous spaces;
4. Grassmannians;
5. Segre and Veronese varieties.
Segre–Veronese Varieties

Consider vector spaces $V_i$, $i = 1, \cdots, n$ with duals $V_i^*$, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P} V_1^* \times \cdots \times \mathbb{P} V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$. 
Segre–Veronese Varieties

Consider vector spaces $V_i$, $i = 1, \cdots, n$ with duals $V_i^*$, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P} V_1^* \times \cdots \times \mathbb{P} V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$. $X$ is the image of

$$SV_{d_1, \cdots, d_n} : \mathbb{P} V_1^* \times \cdots \times \mathbb{P} V_n^* \to \mathbb{P}(\text{Sym}^{d_1} V_1^* \otimes \cdots \otimes \text{Sym}^{d_n} V_n^*),$$

$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call $X$ a Segre–Veronese variety.
Segre–Veronese Varieties

Consider vector spaces $V_i$, $i = 1, \cdots, n$ with duals $V_i^*$, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P} V_1^* \times \cdots \times \mathbb{P} V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$. $X$ is the image of

$$SV_{d_1, \cdots, d_n} : \mathbb{P} V_1^* \times \cdots \times \mathbb{P} V_n^* \to \mathbb{P} (\text{Sym}^{d_1} V_1^* \otimes \cdots \otimes \text{Sym}^{d_n} V_n^*),$$

$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call $X$ a Segre–Veronese variety. Write $W^*$ for the linear forms on the target of $SV_{d_1, \cdots, d_n}$, $W^* = \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_n} V_n$. To compute the equations of $\sigma_k(X)$ it’s “enough” to understand the kernel of

$$s^\# : \text{Sym}(W^*) \to \left( \bigoplus_{r \geq 0} \text{Sym}^{rd_1} V_1 \otimes \cdots \otimes \text{Sym}^{rd_n} V_n \right)^{\otimes k}.$$
Example: generic matrices, flattenings

When all $d_i = 1$, $X$ is the Segre variety (pure tensors). When $n = 2$ we get matrices of rank 1 as the image of

$$SV_{1,1} : \mathbb{P}V_1^* \times \mathbb{P}V_2^* \to \mathbb{P}(V_1^* \otimes V_2^*).$$
Example: generic matrices, flattenings

When all $d_i = 1$, $X$ is the Segre variety (pure tensors). When $n = 2$ we get matrices of rank 1 as the image of

$$SV_{1,1} : \mathbb{P} V_1^* \times \mathbb{P} V_2^* \to \mathbb{P}(V_1^* \otimes V_2^*).$$

More generally, $\sigma_k(X)$ is the collection of matrices of rank at most $k$, which are defined by the vanishing of their $(k + 1)$–minors.
Example: generic matrices, flattenings

When all $d_i = 1$, $X$ is the Segre variety (pure tensors). When $n = 2$ we get matrices of rank 1 as the image of

$$SV_{1,1} : \mathbb{P}V_1^* \times \mathbb{P}V_2^* \to \mathbb{P}(V_1^* \otimes V_2^*).$$

More generally, $\sigma_k(X)$ is the collection of matrices of rank at most $k$, which are defined by the vanishing of their $(k + 1)$–minors.

If $n = 3$, $\dim(V_i) = 3$, the ambient space consists of $3 \times 3 \times 3$ tensors $T = (x_{ijk})$, which we can flatten by thinking of $V_1^* \otimes V_2^*$ as a single factor:

$$V_1^* \otimes V_2^* \otimes V_3^* = (V_1^* \otimes V_2^*) \otimes V_3^*.$$
Example: generic matrices, flattenings

When all $d_i = 1$, $X$ is the Segre variety (pure tensors). When $n = 2$ we get matrices of rank 1 as the image of

$$SV_{1,1} : \mathbb{P} V_1^* \times \mathbb{P} V_2^* \to \mathbb{P}(V_1^* \otimes V_2^*).$$

More generally, $\sigma_k(X)$ is the collection of matrices of rank at most $k$, which are defined by the vanishing of their $(k + 1)$–minors.

If $n = 3$, $\dim(V_i) = 3$, the ambient space consists of $3 \times 3 \times 3$ tensors $T = (x_{ijk})$, which we can flatten by thinking of $V_1^* \otimes V_2^*$ as a single factor:

$$V_1^* \otimes V_2^* \otimes V_3^* = (V_1^* \otimes V_2^*) \otimes V_3^*.$$

The tensor $T$ flattens to a $3 \times 9$ matrix

$$
\begin{bmatrix}
    x_{11,1} & x_{12,1} & x_{13,1} & x_{21,1} & x_{22,1} & x_{23,1} & x_{31,1} & x_{32,1} & x_{33,1} \\
    x_{11,2} & x_{12,2} & x_{13,2} & x_{21,2} & x_{22,2} & x_{23,2} & x_{31,2} & x_{32,2} & x_{33,2} \\
    x_{11,3} & x_{12,3} & x_{13,3} & x_{21,3} & x_{22,3} & x_{23,3} & x_{31,3} & x_{32,3} & x_{33,3}
\end{bmatrix}
$$
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
A conjecture about flattenings

1. \( \sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \): 2–minors of flattenings. (Kostant)
2. \( \sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \): 3–minors of flattenings. (Landsberg–Manivel)
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)

Conjecture (Garcia–Stillman–Sturmfels, Pachter–Sturmfels)

The 3–minors of flattenings generate the ideal of $\sigma_2(X)$ when $X$ is a Segre variety.
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)

Conjecture (Garcia–Stillman–Sturmfels, Pachter–Sturmfels)

The 3–minors of flattenings generate the ideal of $\sigma_2(X)$ when $X$ is a Segre variety.

Known cases:

1. 2 factors (classical)
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)

Conjecture (Garcia–Stillman–Sturmfels, Pachter–Sturmfels)

The 3–minors of flattenings generate the ideal of $\sigma_2(X)$ when $X$ is a Segre variety.

Known cases:

1. 2 factors (classical)
2. 3 factors, and the set–theoretic version for any number of factors (Landsberg and Manivel)
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)

Conjecture (Garcia–Stillman–Sturmfels, Pachter–Sturmfels)

The 3–minors of flattenings generate the ideal of $\sigma_2(X)$ when $X$ is a Segre variety.

Known cases:

1. 2 factors (classical)
2. 3 factors, and the set–theoretic version for any number of factors (Landsberg and Manivel)
3. 4 factors (Landsberg and Weyman)
A conjecture about flattenings

1. $\sigma_1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 2–minors of flattenings. (Kostant)
2. $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 3–minors of flattenings. (Landsberg–Manivel)
3. $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$: 4–minors of flattenings give nothing. Instead, use Strassen’s commutation conditions. (Landsberg–Weyman)

Conjecture (Garcia–Stillman–Sturmfels, Pachter–Sturmfels)

The 3–minors of flattenings generate the ideal of $\sigma_2(X)$ when $X$ is a Segre variety.

Known cases:

1. 2 factors (classical)
2. 3 factors, and the set–theoretic version for any number of factors (Landsberg and Manivel)
3. 4 factors (Landsberg and Weyman)
4. 5 factors (Allman and Rhodes)
Example: Veronese embeddings of $\mathbb{P}^1$

When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis \{x, y\} of $V^*$), $X$ is a rational normal curve of degree $d$, embedded by

$$[x : y] \mapsto [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

Write $z_i \in \text{Sym}^d V$ for the coordinate function of the ambient projective space $\mathbb{P}(\text{Sym}^d V^*)$ corresponding to $x^{d-i} \cdot y^i$. 
Example: Veronese embeddings of $\mathbb{P}^1$

When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis $\{x, y\}$ of $V^*$), $X$ is a rational normal curve of degree $d$, embedded by

$$[x : y] \longrightarrow [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

Write $z_i \in \text{Sym}^d V$ for the coordinate function of the ambient projective space $\mathbb{P}(\text{Sym}^d V^*)$ corresponding to $x^{d-i} \cdot y^i$. We obtain symmetric flattenings (or catalecticant matrices $\text{Cat}(a, b)$) by writing the multiplication table of $\text{Sym}^a V \otimes \text{Sym}^b V \rightarrow \text{Sym}^d V$. 
Example: Veronese embeddings of $\mathbb{P}^1$
When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis $\{x, y\}$ of $V^*$), $X$ is a rational normal curve of degree $d$, embedded by

$$[x : y] \mapsto [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

Write $z_i \in \text{Sym}^d V$ for the coordinate function of the ambient projective space $\mathbb{P}(\text{Sym}^d V^*)$ corresponding to $x^{d-i} \cdot y^i$. We obtain symmetric flattenings (or catalecticant matrices $\text{Cat}(a, b)$) by writing the multiplication table of $\text{Sym}^a V \otimes \text{Sym}^b V \to \text{Sym}^d V$. For $d = 6$, we get

\[
\begin{array}{c|ccccc}
 & x^3 & x^2 \cdot y & x \cdot y^2 & y^3 \\
--- & --- & --- & --- & --- \\
x^3 & z_0 & z_1 & z_2 & z_3 \\
x^2 \cdot y & z_1 & z_2 & z_3 & z_4 \\
x \cdot y^2 & z_2 & z_3 & z_4 & z_5 \\
y^3 & z_3 & z_4 & z_5 & z_6 \\
\end{array}
\]
Example: Veronese embeddings of $\mathbb{P}^1$

When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis $\{x, y\}$ of $V^*$), $X$ is a rational normal curve of degree $d$, embedded by

$$[x : y] \mapsto [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

Write $z_i \in \text{Sym}^d V$ for the coordinate function of the ambient projective space $\mathbb{P}(\text{Sym}^d V^*)$ corresponding to $x^{d-i} \cdot y^i$. We obtain symmetric flattenings (or catalecticant matrices $\text{Cat}(a, b)$) by writing the multiplication table of $\text{Sym}^a V \otimes \text{Sym}^b V \to \text{Sym}^d V$. For $d = 6$, we get

\[
\begin{array}{c|cccc}
 & x^3 & x^2 \cdot y & x \cdot y^2 & y^3 \\
\hline
x^3 & z_0 & z_1 & z_2 & z_3 \\
x^2 \cdot y & z_1 & z_2 & z_3 & z_4 \\
x \cdot y^2 & z_2 & z_3 & z_4 & z_5 \\
y^3 & z_3 & z_4 & z_5 & z_6 \\
\end{array}
\]

\[
\text{Cat}(3, 3):
\[
\begin{bmatrix}
z_0 & z_1 & z_2 & z_3 & z_4 \\
z_1 & z_2 & z_3 & z_4 & z_5 \\
z_2 & z_3 & z_4 & z_5 & z_6 \\
\end{bmatrix}
\]

\[
\text{Cat}(2, 4):
\[
\begin{bmatrix}
z_0 & z_1 & z_2 & z_3 & z_4 \\
z_1 & z_2 & z_3 & z_4 & z_5 \\
z_2 & z_3 & z_4 & z_5 & z_6 \\
\end{bmatrix}
\]
Example: Veronese embeddings of $\mathbb{P}^1$

When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis $\{x, y\}$ of $V^*$), $X$ is a rational normal curve of degree $d$, embedded by

$$[x : y] \mapsto [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

Write $z_i \in \text{Sym}^d V$ for the coordinate function of the ambient projective space $\mathbb{P}(\text{Sym}^d V^*)$ corresponding to $x^{d-i} \cdot y^i$. We obtain symmetric flattenings (or catalecticant matrices $\text{Cat}(a, b)$) by writing the multiplication table of $\text{Sym}^a V \otimes \text{Sym}^b V \to \text{Sym}^d V$. For $d = 6$, we get

**Cat(3, 3):**

<table>
<thead>
<tr>
<th>$x^3$</th>
<th>$x^2 \cdot y$</th>
<th>$x \cdot y^2$</th>
<th>$y^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>$z_0$</td>
<td>$z_1$</td>
<td>$z_2$</td>
</tr>
<tr>
<td>$x^2 \cdot y$</td>
<td>$z_1$</td>
<td>$z_2$</td>
<td>$z_3$</td>
</tr>
<tr>
<td>$x \cdot y^2$</td>
<td>$z_2$</td>
<td>$z_3$</td>
<td>$z_4$</td>
</tr>
<tr>
<td>$y^3$</td>
<td>$z_3$</td>
<td>$z_4$</td>
<td>$z_5$</td>
</tr>
</tbody>
</table>

**Cat(5, 1):**

<table>
<thead>
<tr>
<th>$z_0$</th>
<th>$z_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1$</td>
<td>$z_2$</td>
</tr>
<tr>
<td>$z_2$</td>
<td>$z_3$</td>
</tr>
<tr>
<td>$z_3$</td>
<td>$z_4$</td>
</tr>
<tr>
<td>$z_4$</td>
<td>$z_5$</td>
</tr>
<tr>
<td>$z_5$</td>
<td>$z_6$</td>
</tr>
</tbody>
</table>
Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If \( X \) is a rational normal curve of degree \( d \), then \( I(\sigma_k(X)) \) is generated by the \((k + 1)\)–minors of any \( \text{Cat}(a, b) \), where \( a, b \geq k \), \( a + b = d \).
Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k + 1)$–minors of any $Cat(a, b)$, where $a, b \geq k, a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $Cat(a, b)$ whenever $a + b = d$. 

Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k + 1)$–minors of any $\text{Cat}(a, b)$, where $a, b \geq k$, $a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $\text{Cat}(a, b)$ whenever $a + b = d$.

1. $(k + 1)$–minors of catalecticants vanish on $\sigma_k(X)$.
Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k + 1)$–minors of any $\text{Cat}(a, b)$, where $a, b \geq k$, $a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $\text{Cat}(a, b)$ whenever $a + b = d$.

1. $(k + 1)$–minors of catalecticants vanish on $\sigma_k(X)$.
2. $X = \sigma_1(X)$ is defined by the 2–minors of any $\text{Cat}(a, b)$. (Pucci)
Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k+1)$–minors of any $Cat(a, b)$, where $a, b \geq k$, $a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $Cat(a, b)$ whenever $a + b = d$.

1. $(k+1)$–minors of catalecticants vanish on $\sigma_k(X)$.
2. $X = \sigma_1(X)$ is defined by the 2–minors of any $Cat(a, b)$. (Pucci)
3. $\sigma_2(X)$ is defined by the 3–minors of $Cat(1, d – 1)$ and $Cat(2, d – 2)$. (Kanev)
Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k + 1)$–minors of any $\text{Cat}(a, b)$, where $a, b \geq k$, $a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $\text{Cat}(a, b)$ whenever $a + b = d$.

1. $(k + 1)$–minors of catalecticants vanish on $\sigma_k(X)$.
2. $X = \sigma_1(X)$ is defined by the 2–minors of any $\text{Cat}(a, b)$. (Pucci)
3. $\sigma_2(X)$ is defined by the 3–minors of $\text{Cat}(1, d − 1)$ and $\text{Cat}(2, d − 2)$. (Kanev)
4. $\sigma_k(X)$ is NOT defined by $(k + 1)$–minors of catalecticants in general. (Buczyńska–Buczyński)
Veronese varieties

**Theorem (Gruson–Peskine, Eisenbud, Conca)**

If $X$ is a rational normal curve of degree $d$, then $I(\sigma_k(X))$ is generated by the $(k + 1)$–minors of any $\text{Cat}(a, b)$, where $a, b \geq k$, $a + b = d$.

Now assume that $\dim(V)$ is arbitrary. We can still talk about catalecticant matrices $\text{Cat}(a, b)$ whenever $a + b = d$.

1. $(k + 1)$–minors of catalecticants vanish on $\sigma_k(X)$.
2. $X = \sigma_1(X)$ is defined by the 2–minors of any $\text{Cat}(a, b)$. (Pucci)
3. $\sigma_2(X)$ is defined by the 3–minors of $\text{Cat}(1, d - 1)$ and $\text{Cat}(2, d - 2)$. (Kanev)
4. $\sigma_k(X)$ is NOT defined by $(k + 1)$–minors of catalecticants in general. (Buczyńska–Buczyński)

**Conjecture (Geramita)**

*The ideals of 3–minors of $\text{Cat}(a, b)$ are all equal for $a, b \geq 2$.***
Main Results

Theorem (–)

Geramita conjecture holds, as well as its generalization to 4–minors.
Main Results

Theorem (−)
Geramita conjecture holds, as well as its generalization to 4–minors.

Question
Are the ideals of $k$–minors of $\text{Cat}(a, b)$ all equal for $a, b \geq k - 1$?
Main Results

Theorem (—)
Geramita conjecture holds, as well as its generalization to 4–minors.

Question
Are the ideals of k–minors of Cat(a, b) all equal for a, b \geq k – 1?

Theorem (—)
For X a Segre–Veronese variety, the ideal of \( \sigma_2(X) \) is generated by 3–minors of flattenings. Moreover, one has an explicit description of the multiplicities of the irreducible representations that occur in the decomposition of the homogeneous coordinate ring of \( \sigma_2(X) \).
Suppose \( n = 2, \, d_1 = 2, \, d_2 = 1 \), and focus on the equations of degree 4 of \( \sigma_2(X) \). We look for the kernel of

\[
\sigma^* : \text{Sym}^4(\text{Sym}^2 \, V_1 \otimes V_2) \rightarrow \bigoplus_{a+b=4} (\text{Sym}^{2a} \, V_1 \otimes \text{Sym}^a \, V_2) \otimes (\text{Sym}^{2b} \, V_1 \otimes \text{Sym}^b \, V_2).
\]
Suppose $n = 2$, $d_1 = 2$, $d_2 = 1$, and focus on the equations of degree 4 of $\sigma_2(X)$. We look for the kernel of

$$s^\# : \text{Sym}^4(\text{Sym}^2 V_1 \otimes V_2) \rightarrow \bigoplus_{a+b=4} (\text{Sym}^{2a} V_1 \otimes \text{Sym}^{a} V_2) \otimes (\text{Sym}^{2b} V_1 \otimes \text{Sym}^{b} V_2).$$

“Representation theory yoga” $\Rightarrow$ free to choose $m_i = \text{dim}(V_i)$ arbitrarily, as long as $m_i \geq 2$. 
Polarization and Specialization

Suppose $n = 2$, $d_1 = 2$, $d_2 = 1$, and focus on the equations of degree 4 of $\sigma_2(X)$. We look for the kernel of

$$s^\#: \text{Sym}^4(\text{Sym}^2 V_1 \otimes V_2) \longrightarrow \bigoplus_{a+b=4} (\text{Sym}^{2a} V_1 \otimes \text{Sym}^a V_2) \otimes (\text{Sym}^{2b} V_1 \otimes \text{Sym}^b V_2).$$

“Representation theory yoga” $\implies$ free to choose $m_i = \dim(V_i)$ arbitrarily, as long as $m_i \geq 2$. Take $m_1 = 8$, $m_2 = 4$. 
Polarization and Specialization

Suppose $n = 2$, $d_1 = 2$, $d_2 = 1$, and focus on the equations of degree 4 of $\sigma_2(X)$. We look for the kernel of

$$s^\# : \text{Sym}^4(\text{Sym}^2 V_1 \otimes V_2) \longrightarrow \bigoplus_{a+b=4} (\text{Sym}^{2a} V_1 \otimes \text{Sym}^a V_2) \otimes (\text{Sym}^{2b} V_1 \otimes \text{Sym}^b V_2).$$

“Representation theory yoga” $\Rightarrow$ free to choose $m_i = \dim(V_i)$ arbitrarily, as long as $m_i \geq 2$. Take $m_1 = 8$, $m_2 = 4$. The (SL-) zero–weight spaces $S$ and $T$ of the source and target of $s^\#$ are representations of the Weyl group $S_8 \times S_4$. Enough to analyze

$$s_0^\# : S \longrightarrow T.$$
Suppose \( n = 2, d_1 = 2, d_2 = 1 \), and focus on the equations of degree 4 of \( \sigma_2(X) \). We look for the kernel of

\[
\begin{align*}
\sigma^\# : \text{Sym}^4(\text{Sym}^2 V_1 \otimes V_2) & \longrightarrow \\
\bigoplus_{a + b = 4} (\text{Sym}^{2a} V_1 \otimes \text{Sym}^a V_2) \otimes (\text{Sym}^{2b} V_1 \otimes \text{Sym}^b V_2).
\end{align*}
\]

"Representation theory yoga" \( \Rightarrow \) free to choose \( m_i = \dim(V_i) \) arbitrarily, as long as \( m_i \geq 2 \). Take \( m_1 = 8, m_2 = 4 \). The (SL-) zero–weight spaces \( S \) and \( T \) of the source and target of \( \sigma^\# \) are representations of the Weyl group \( S_8 \times S_4 \). Enough to analyze

\[
s_0^\# : S \longrightarrow T.
\]

To do that, use the representation theory of (products of) symmetric groups, and the combinatorics that comes with it.
Polarization and Specialization

A typical monomial in $S$ looks like

$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

$((x_i)_i$ and $(y_j)_j$ are bases for $V_1, V_2$).
Polarization and Specialization

A typical monomial in $S$ looks like

$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

$$(x_i)_i \text{ and } (y_j)_j \text{ are bases for } V_1, V_2). \text{ It specializes to }$$

$$m = (x_1^2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_2 x_3 \otimes y_2) \cdot (x_3^2 \otimes y_2)$$

via the specialization map $\phi$ that sends

$$\{x_1, x_2\} \to x_1, \{x_4, x_6\} \to x_2, \{x_3, x_5, x_7, x_8\} \to x_3,$$

$$\{y_1\} \to y_1, \{y_2, y_3, y_4\} \to y_2.$$
Polarization and Specialization

A typical monomial in $S$ looks like

$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

$$(x_i)_i \text{ and } (y_j)_j \text{ are bases for } V_1, V_2). \text{ It specializes to }$ 

$$m = (x_1^2 \otimes y_2) \cdot (x_3 x_2 \otimes y_1) \cdot (x_2 x_3 \otimes y_2) \cdot (x_3^2 \otimes y_2)$$

via the specialization map $\phi$ that sends

$$\{x_1, x_2\} \rightarrow x_1, \quad \{x_4, x_6\} \rightarrow x_2, \quad \{x_3, x_5, x_7, x_8\} \rightarrow x_3,$$

$$\{y_1\} \rightarrow y_1, \quad \{y_2, y_3, y_4\} \rightarrow y_2.$$

Any kernel element of $s_0^\#$ specializes to a kernel element of $s^\#$. 
Polarization and Specialization

A typical monomial in $S$ looks like

$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

$((x_i)_i$ and $(y_j)_j$ are bases for $V_1, V_2$). It specializes to

$$m = (x_1^2 \otimes y_2) \cdot (x_3 x_2 \otimes y_1) \cdot (x_2 x_3 \otimes y_2) \cdot (x_3^2 \otimes y_2)$$

via the specialization map $\phi$ that sends

$$\{x_1, x_2\} \to x_1, \quad \{x_4, x_6\} \to x_2, \quad \{x_3, x_5, x_7, x_8\} \to x_3,$$

$$\{y_1\} \to y_1, \quad \{y_2, y_3, y_4\} \to y_2.$$

Any kernel element of $s_0^#$ specializes to a kernel element of $s^#$. We can polarize $m$ by

$$m \mapsto \text{average}(m_0 : \phi(m_0) = m).$$
Polarization and Specialization

A typical monomial in $S$ looks like

$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

$((x_i)_i$ and $(y_j)_j$ are bases for $V_1, V_2)$. It specializes to

$$m = (x_1^2 \otimes y_2) \cdot (x_3 x_2 \otimes y_1) \cdot (x_2 x_3 \otimes y_2) \cdot (x_3^2 \otimes y_2)$$

via the specialization map $\phi$ that sends

$$\{x_1, x_2\} \rightarrow x_1, \quad \{x_4, x_6\} \rightarrow x_2, \quad \{x_3, x_5, x_7, x_8\} \rightarrow x_3,$$

$$\{y_1\} \rightarrow y_1, \quad \{y_2, y_3, y_4\} \rightarrow y_2.$$

Any kernel element of $s^\#_0$ specializes to a kernel element of $s^\#$. We can polarize $m$ by

$$m \mapsto \text{average}(m_0 : \phi(m_0) = m).$$

Any kernel element of $s^\#$ polarizes to a kernel element of $s^\#_0$. 