Secant Varieties of Segre–Veronese Varieties

Claudiu Raicu

Princeton University

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Overview

Secant Varieties

Definition

Given a subvariety $X \subset \mathbb{P}^N$, the $(k-1)$–st secant variety of X, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by k points on X:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\cdots,x_k \in X} \mathbb{P}_{x_1,\cdots,x_k}}.$$

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Alternatively, write $\mathbb{P}^N = \mathbb{P}W$ for some vector space W, and let $\hat{X} \subset W$ denote the cone over X. The cone $\widehat{\sigma_k(X)}$ over $\sigma_k(X)$ is the closure of the image of the map

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Problem

Given (the equations of) X, determine (the equations of) $\sigma_k(X)$.

Solution to Problem

The morphism s of affine varieties corresponds to a ring map

$$s^{\#} : \mathrm{Sym}(W^*) \to K[X \times \cdots \times X] = K[X] \otimes \cdots \otimes K[X].$$

$I(\sigma_k(X))$ and $K[\sigma_k(X)]$ are the kernel and image respectively of $s^{\#}$.

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5. Segre and Veronese varieties.

Segre–Veronese Varieties

Consider vector spaces V_i, $i = 1, \cdots, n$ with duals V_i^*, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$.

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$$SV_{d_1,\cdots,d_n} : \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^* \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*),$$

$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call X a Segre–Veronese variety.

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$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call X a Segre–Veronese variety. Write W^* for the linear forms on the target of $SV_{d_1,\cdots,d_n}$, $W^* = \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n$. To compute the equations of $\sigma_k(X)$ it's "enough" to understand the kernel of

$$s^\# : \mathrm{Sym}(W^*) \longrightarrow \left(\bigoplus_{r \geq 0} \mathrm{Sym}^{rd_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{rd_n} V_n \right)^{\otimes k}.$$

Example: generic matrices, flattenings

When all $d_i = 1$, X is the Segre variety (pure tensors). When $n = 2$ we get matrices of rank 1 as the image of

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If $n = 3$, $\dim(V_i) = 3$, the ambient space consists of $3 \times 3 \times 3$ tensors $T = (x_{ijk})$, which we can flatten by thinking of $V_1^* \otimes V_2^*$ as a single factor:

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The tensor T flattens to a 3×9 matrix

$$\left[\begin{array}{ccc|ccc|ccc} x_{11,1} & x_{12,1} & x_{13,1} & x_{21,1} & x_{22,1} & x_{23,1} & x_{31,1} & x_{32,1} & x_{33,1} \\ x_{11,2} & x_{12,2} & x_{13,2} & x_{21,2} & x_{22,2} & x_{23,2} & x_{31,2} & x_{32,2} & x_{33,2} \\ x_{11,3} & x_{12,3} & x_{13,3} & x_{21,3} & x_{22,3} & x_{23,3} & x_{31,3} & x_{32,3} & x_{33,3} \end{array}\right]$$

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4. 5 factors (Allman and Rhodes)

Example: Veronese embeddings of $\mathbb{P}^1$

When $n = 1$, write $V = V_1$, $d = d_1$. If $\dim(V) = 2$ (with basis $\{x, y\}$ of V^*), X is a rational normal curve of degree d, embedded by

$$[x : y] \longrightarrow [x^d : x^{d-1} \cdot y : \cdots : x \cdot y^{d-1} : y^d].$$

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$Cat(3,3)$:

	x^3	$x^2 \cdot y$	$x \cdot y^2$	y^3
x^3	z_0	z_1	z_2	z_3
$x^2 \cdot y$	z_1	z_2	z_3	z_4
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Veronese varieties

Theorem (Gruson–Peskine, Eisenbud, Conca)

If X is a rational normal curve of degree d, then $I(\sigma_k(X))$ is generated by the $(k+1)$–minors of any $Cat(a,b)$, where $a,b \geq k$, $a+b=d$.

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Conjecture (Geramita)

The ideals of 3–minors of $Cat(a,b)$ are all equal for $a,b \geq 2$.

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Question

Are the ideals of k–minors of $Cat(a, b)$ all equal for $a, b \geq k - 1$?

Main Results

Theorem (–)

Geramita conjecture holds, as well as its generalization to 4–minors.

Question

Are the ideals of k–minors of $Cat(a, b)$ *all equal for* $a, b \geq k - 1$*?*

Theorem (–)

For X *a Segre–Veronese variety, the ideal of* $\sigma_2(X)$ *is generated by 3–minors of flattenings. Moreover, one has an explicit description of the multiplicities of the irreducible representations that occur in the decomposition of the homogeneous coordinate ring of* $\sigma_2(X)$*.*

Polarization and Specialization

Suppose $n = 2$, $d_1 = 2$, $d_2 = 1$, and focus on the equations of degree 4 of $\sigma_2(X)$. We look for the kernel of

$$s^{\#} : \mathrm{Sym}^4(\mathrm{Sym}^2 V_1 \otimes V_2) \longrightarrow$$

$$\bigoplus_{a+b=4} (\mathrm{Sym}^{2a} V_1 \otimes \mathrm{Sym}^{a} V_2) \otimes (\mathrm{Sym}^{2b} V_1 \otimes \mathrm{Sym}^{b} V_2).$$

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To do that, use the representation theory of (products of) symmetric groups, and the combinatorics that comes with it.

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$$(x_1 x_2 \otimes y_2) \cdot (x_3 x_6 \otimes y_1) \cdot (x_4 x_7 \otimes y_4) \cdot (x_5 x_8 \otimes y_3),$$

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$$m = (x_1^2 \otimes y_2) \cdot (x_3x_2 \otimes y_1) \cdot (x_2x_3 \otimes y_2) \cdot (x_3^2 \otimes y_2)$$

via the specialization map ϕ that sends

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