Tangential Varieties of Segre–Veronese Varieties

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Theorem (Fulton–Hansen '79)

Precisely one of the following holds:
(i) $\dim(Tan(X)) = 2n$ *and* $\dim(Sec(X)) = 2n + 1$*; (typical)*
(ii) $Tan(X) = Sec(X)$*. (degenerate)*

Example 1: rank one matrices

Consider vector spaces V_1, V_2, $\dim(V_i) = n_i$, and the Segre variety

$$X = \mathrm{Im}\left(\mathbb{P}V_1 \times \mathbb{P}V_2 \xrightarrow{\mathcal{O}(1,1)} \mathbb{P}(V_1 \otimes V_2)\right),$$

which we can think of as the set of indecomposable bilinear forms $f_1 \otimes f_2 \in V_1^* \otimes V_2^*$, or as the space of rank one $n_1 \times n_2$ matrices.

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Typical elements of the secant and tangential varieties of X look like:

$$Sec(X) : f_1 \otimes f_2 + g_1 \otimes g_2;$$

$$Tan(X) : \lim_{t \to 0} \frac{(f_1 + tg_1) \otimes (f_2 + tg_2) - f_1 \otimes f_2}{t} = f_1 \otimes g_2 + g_1 \otimes f_2.$$

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It follows that X is degenerate. Furthermore, $Sec(X) = Tan(X)$ has dimension $2 \cdot (n_1 + n_2 - 2) - 1$, and is defined by the vanishing of the 3×3 minors of the generic $n_1 \times n_2$ matrix.

Example 2: rank one symmetric matrices

Consider a vector space V, $\dim(V) = n$, and the quadratic Veronese variety

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In both cases we get the Chow variety of quadratic forms that decompose into a product of linear factors, so X is again degenerate. Furthermore, $Sec(X) = Tan(X)$ has dimension $2 \cdot (n - 1)$, and is defined by the vanishing of the 3×3 minors of the generic $n \times n$ symmetric matrix.

Abo–Brambilla Theorem

Consider vector spaces V_i, $i = 1, \cdots, n$, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$. X is the image of

$$SV_{d_1,\cdots,d_n} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n),$$

$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call X a Segre–Veronese variety.

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Theorem (Abo–Brambilla '09)

If X is a Segre–Veronese variety, then X is degenerate if and only if X is as in Examples 1 and 2.

Example 3: the twisted cubic

Consider the twisted cubic

$$X = \{f^3 : f \text{ linear form in } x, y\} \subset \mathbb{P}^3 = \{ax^3 + bx^2y + cxy^2 + dy^3\}.$$

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It follows that

$$Tan(X) = \{f^2 g : f, g \text{ linear forms}\},$$

i.e. the tangential variety is given by the cubics with a double root. It is thus a quartic surface, defined by the vanishing of the discriminant

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One can check that

$$Sec(X) = \mathbb{P}^3,$$

so the twisted cubic is typical.

Two conjectures regarding Segre varieties

Conjecture (Garcia–Stillman–Sturmfels '05)

If X is a Segre variety, then $I(Sec(X))$ is generated by the 3×3 minors of certain matrices of linear forms (called flattenings).

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If X is a Segre–Veronese variety, then all the minimal generators of $I(Sec(X))$ have degree 3 (they are 3×3 minors of flattenings).

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- *minimal generators of degree 2 occur if and only if $\sum_{i=1}^n d_i \geq 4$.*
- *$I(Tan(X))$ is generated by quadrics if and only if $X = SV_{d,1}(\mathbb{P}^1 \times \mathbb{P}^r)$ for $d \geq 5$.*

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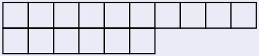
- $I(Sec(X))$ has only minimal generators of degree 3.
- $I(Tan(X))$ has minimal generators of degree $\begin{cases} 2 & \text{if } \sum d_i \geq 4; \\ 4 & \text{if } \sum d_i = 3. \end{cases}$

It follows that if $Sec(X) = Tan(X)$ then $\sum d_i = 2$, i.e. we are in the situation of Example 1 or 2.

Tangential of a Veronese variety

Theorem (O– '13)

Let $d \geq 2$, and let $X = SV_d(\mathbb{P}V)$ be a Veronese variety. The module of minimal generators of $I(Tan(X))$ of degree q decomposes as $\bigoplus_\lambda (S_\lambda V)^{\oplus m_\lambda}$, where $m_\lambda \in \{0, 1\}$, with $m_\lambda = 1$ precisely in the following cases:

1. $q = 2$: $\lambda = (2d - k, k)$ for $4 \leq k \leq d$, k even: 

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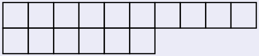
1. $q = 2$: $\lambda = (2d-k, k)$ for $4 \leq k \leq d$, k even: 
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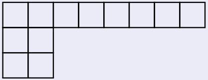
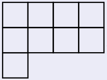
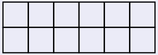
 ,  , *resp.* 

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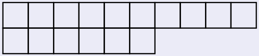
1. $q = 2$: $\lambda = (2d - k, k)$ *for* $4 \leq k \leq d$, k *even*: 
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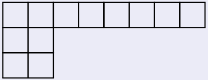
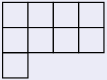
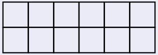
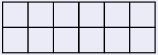
    ,  , *resp.* 

3. $q = 4$: $\lambda = (6, 6)$ *when* $d = 3$: 

The coordinate ring of the tangential

Theorem (O– '13)

Let $X = SV_{d_1,\cdots,d_n}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ be a Segre–Veronese variety. The degree r part of the homogeneous coordinate ring of $Tan(X)$ decomposes as

$$\mathbb{K}[Tan(X)]_r = \bigoplus_{\substack{\lambda=(\lambda^1,\cdots,\lambda^n) \\ \lambda^j \vdash rd_j}} (S_{\lambda^1} V_1 \otimes \cdots \otimes S_{\lambda^n} V_n)^{\oplus m_\lambda},$$

where m_λ is either 0 or 1, obtained as follows. Set

$$f_\lambda = \max_{j=1,\cdots,n} \left\lceil \frac{\lambda_2^j}{d_j} \right\rceil, \quad e_\lambda = \lambda_2^1 + \cdots + \lambda_2^n.$$

If some λ^j has more than two parts, or $e_\lambda < 2f_\lambda$, or $e_\lambda > r$, then $m_\lambda = 0$, else $m_\lambda = 1$.