Secant Varieties of Segre-Veronese Varieties

by

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Abstract

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Secant varieties of Segre and Veronese varieties (and more generally Segre-Veronese varieties, which are embeddings of a product of projective spaces via the complete linear system of an ample line bundle) are very classical objects that go back to the Italian school of mathematics in the 19-th century. Despite their apparent simplicity, little is known about their equations, and even less about the resolutions of their coordinate rings. The main goal of this thesis is to introduce a new method for analyzing the equations and coordinate rings of the secant varieties to Segre-Veronese varieties, and to work out the details of this method in the first case of interest: the variety of secant lines to a Segre-Veronese variety.

There is an extensive literature explaining the advantages of analyzing the equations of the secant varieties of $X \subset \mathbb{P}^N$ as G-modules, when X is endowed with a G-action that extends to $\mathbb{P}^N$. For X a Segre-Veronese variety, the corresponding G is a general linear (GL) group, or a product of such. Looking inside the highest weight spaces of carefully chosen GL-representations, we identify a set of "generic equations" for the secant varieties of Segre-Veronese varieties. The collections of "generic equations" form naturally modules over (products of) symmetric groups and moreover, they yield by the process of specialization all the (nongeneric) equations of the secant varieties of Segre-Veronese varieties.

Once we reduce our problem to the analysis of "generic equations", the representation theory of symmetric groups comes into play, and with it the combinatorics of tableaux. In the case of the first secant variety of a Segre-Veronese variety, we are naturally led to consider 1-dimensional simplicial complexes, i.e. graphs, attached to the relevant tableaux. We believe that simplicial complexes should play an important role in the combinatorics that emerges in the case of higher secant varieties.

The results of this thesis go in two directions. For both of them, the reduction to the "generic" situation is used in an essential way. One direction is showing that if we put together the 3×3 minors of certain generic matrices (called flattenings), we obtain a generating set for the ideal of the secant line variety of a Segre-Veronese variety. In particular, this recovers a conjecture of Garcia, Stillman and Sturmfels, corresponding to the case of

a Segre variety. We also give a representation theoretic description of the homogeneous coordinate ring of the secant line variety of a Segre-Veronese variety. In the cases when this secant variety fills the ambient space, we obtain formulas for decomposing certain plethystic compositions.

A different direction is, for the Veronese variety, to show that for k small, the ideal of $k \times k$ minors of the various flattenings (which in this case are also known as catalecticant matrices) are essentially independent of which flattening we choose. In particular this proves a conjecture of Geramita, stating that the ideals of 3×3 minors of the "middle" catalecticant matrices are the same, and moreover that the ideal of the first secant variety of a Veronese variety is generated by the 3×3 minors of any such catalecticant.

To my family

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Chapter 1

Introduction

1.1 Overview

The study of tensors and particularly tensor ranks is important in numerous fields of research today: algebraic statistics ([GSS05]), biology ([AR08], [PS04]), signal processing ([WvB10]), complexity theory ([Lan08], [BLMW09]). For example, people in computer science would very much like to know how expensive matrix multiplication is - how fast algorithms can be developed for multiplying arbitrary size matrices. The usual assumptions in this problem are that

- *addition* is free;
- *multiplication* has some positive cost, say 1 cent per scalar multiplication.

One could start with the basic

Question 1.1.1. How much does it cost to compute the product of two 2×2 matrices?

The naive answer, based on the usual algorithm for matrix multiplication that we learn in school, would be 8 cents. But in fact one could do better than that, and Strassen's algorithm ([CLRS01], Section 28.2, or [Lan08], Section 1.1) gives a way to multiply two such matrices at a cost of only 7 cents.

More generally, one could ask

Question 1.1.2. What is the cost of multiplying two $n \times n$ matrices, or even more generally, two arbitrary rectangular matrices?

The answer to this is not known in general, and is controlled by the rank of a certain 3-tensor, the tensor of matrix multiplication. If we denote by $M_{a,b}$ the space of $a \times b$ matrices, then matrix multiplication is a linear map

$$T_{m,n,p} : M_{m,n} \otimes M_{n,p} \longrightarrow M_{m,p},$$

i.e. a 3-tensor in $M_{m,n} \otimes M_{n,p} \otimes M_{m,p}^*$. The **rank** of $T_{m,n,p}$ is the minimal possible number of terms in an expression of $T_{m,n,p}$ as a linear combination of pure tensors $A \otimes B \otimes C$.

It turns out that for algebraic geometry, a better suited question is

Question 1.1.3. What is the cost of computing the product of two matrices with arbitrarily high precision?

The reason for this is that it allows one to replace the tensor rank of $T_{m,n,p}$ with its **border rank**, i.e. the minimal number k such that $T_{m,n,p}$ is a limit of tensors of rank k. This replaces the study of certain quasi-projective varieties defined by rank conditions, with that of closed subvarieties in projective space defined by border rank conditions - the secant varieties of triple Segre products. Knowing the equations of such varieties would allow us to evaluate the border rank of any given tensor, in particular that of matrix multiplication, and therefore enable us to answer Question 1.1.3.

It is worth mentioning that not a single equation is known for $\sigma_6(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$, the variety of secant 5-planes to a Segre product of three projective 3-spaces, or equivalently, the space of tensors of border rank at most 6 living in the tensor product of three vector spaces of dimension 4 ($M_{2,2}, M_{2,2}$ and $M_{2,2}^*$). The reason why this would be interesting is because having one such equation that didn't vanish on $T_{2,2,2}$ would allow us to conclude that the border rank of multiplication of 2×2 matrices is at least 7, and hence equal to 7 since it is bounded above by $\mathrm{rank}(T_{2,2,2})$, which by Strassen's algorithm is at most 7.

One of the main goals of this thesis is to set up a general framework that would allow one to attack questions like the ones presented above, and illustrate how insights from combinatorics naturally occur in this framework, providing new results in the case of small secant varieties. The rest of this chapter contains an introduction to the study of the defining equations of secant varieties of Segre-Veronese varieties, as well as a description of catalecticant varieties and their connections to secant varieties and other areas of mathematics. We state our main results and mention the relationship to the existing literature on the subject.

In Chapter 2 we give the basic definitions for secant varieties, Segre-Veronese varieties, catalecticant varieties. We introduce the basic notions from Representation Theory that are used throughout the work, and describe the process of flattening a tensor, which leads to the notion of a flattening matrix. We give a flavour of the problem of analyzing the ideals and coordinate rings of secant varieties of Segre-Veronese varieties, by illustrating the classical case of the zeroth secant variety, the Segre-Veronese variety itself. Our main result in Chapter 4 will describe a completely analogous picture in the case of the first secant variety, the variety of secant lines to a Segre-Veronese variety.

Chapter 3 builds the framework for analyzing the equations and homogeneous coordinate rings of arbitrary secant varieties of Segre-Veronese varieties. Even though we were only able to work out the details of this analysis in the case of the first secant variety, we believe that the general method of approach may be used to shed some light on the case of higher secant varieties. In particular, the new insight of concentrating on the "generic equations" is presented in detail and in the generality needed to deal with arbitrary secant varieties.

Chapter 4 is inspired by a conjecture of Garcia, Stillman and Sturmfels, describing the generators of the ideal of the variety of secant lines to a Segre variety. We prove more generally that this description holds for the first secant variety of a Segre-Veronese variety. We also give a representation theoretic decomposition of the coordinate ring of this variety, which allows us to deduce certain plethystic formulas based on known computations of dimensions of secant varieties of Segre varieties.

In Chapter 5 we describe our combinatorial techniques through a series of examples. We recover the classical Cauchy formula for decomposing symmetric powers of tensor products of two vector spaces. We also describe our view on Strassen's equations from the "generic" perspective, as well as on their generalized version introduced by Landsberg and Manivel ([LM08]). The new result in this chapter is a strengthening of [LM08, Theorem 4.2], by removing an unnecessary assumption, as suggested by the authors. We also show how our methods can be used to recover some of the known equations for the variety of secant 3-planes to a Segre product of three projective 3-spaces - determining the defining ideal of this variety is known under the name of "The Salmon Problem". Finally, we give the "generic" version of the Aronhold invariant, a module of equations for the variety of secant planes to the 3-uple embedding of projective space, that does not come from flattenings.

Finally, Chapter 6 deals with catalecticant varieties, i.e. varieties defined by vanishing of minors of catalecticant matrices. These matrices appear earlier in the work in connection with the equations of the secant varieties of Veronese varieties: they are special cases of matrices of flattenings, but their beauty lies in their ubiquitous nature - determinantal loci of catalecticant matrices are not only related to secant varieties, but also to Hilbert functions of Gorenstein Artin algebras, the polynomial Waring problem, or configurations of points in projective space (see [Ger96], [IK99]). We show that for small values of k and fixed d, most catalecticant matrices arising from flattening the generic symmetric tensor of degree d have the same ideals of $k \times k$ minors. This proves a conjecture of Geramita, and establishes the first new case of its natural generalization.

1.2 Secant Varieties of Segre-Veronese Varieties

The spaces of matrices (or 2-tensors) are stratified according to rank by the secant varieties of Segre products of two projective spaces. The defining ideals of these secant varieties are known to be generated by minors of generic matrices. As mentioned in the previous section, it is an important problem, with numerous applications, to understand (border) rank varieties of higher order tensors. These are (upon taking closure) the classical secant varieties to Segre varieties, whose equations are far from being understood. To get an idea about the boundary of our knowledge, note that the famous Salmon problem ([All]), which asks for the generators of the ideal of $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, the variety of secant 3-planes to the Segre product of three projective 3-spaces, is still unsolved (although its set-theoretic version has been recently resolved in [Fri10, FG11]; see also [BO10]).

Flattenings (see Section 2.5) provide an easy tool for obtaining some equations for secant varieties of Segre products, but they are not sufficient in general, as can be seen for example in the case of the Salmon problem. Inspired by the study of Bayesian networks, Garcia, Stillman and Sturmfels conjectured ([GSS05]) that flattenings give all the equations of the first secant variety of the Segre variety. This conjecture also appeared at the same time in a biological context, namely in work of Pachter and Sturmfels on phylogenetic inference ([PS04], Conjecture 13).

Conjecture 1.2.1 (Garcia-Stillman-Sturmfels)**.** *The ideal of the secant line variety of a Segre product of projective spaces is generated by* 3×3 *minors of flattenings.*

The set-theoretic version of this conjecture was obtained by Landsberg and Manivel ([LM04]), as well as the case of a 3-factor Segre product. The 2-factor case is classical, while the 4-factor case was resolved by Landsberg and Weyman ([LW07]). The 5-factor case was proved by Allman and Rhodes ([AR08]). We prove the GSS conjecture in Corollary 4.1.2 as a consequence of our main result (Theorem 4.1.1).

It is a general fact that for a subvariety X in projective space which is not contained in a hyperplane, the ideal of the variety $\sigma_k(X)$ of secant $(k-1)$-planes to X has no equations in degree less than $k+1$. If $X = G/P$ is a rational homogeneous variety, a theorem of Kostant (see [Lan]) states that the ideal of X is generated in the smallest possible degree (i.e. in degree two), and Landsberg and Manivel asked whether this is also true for the first secant variety to X ([LM04]). It turns out that when X is the D_7-spinor variety, there are in fact no cubics in the ideal of $\sigma_2(X)$ (see [LW09] and [Man09]). In Theorem 4.1.1, we provide a family of G/P's, the Segre-Veronese varieties for which the answer to the question of Landsberg and Manivel is positive. We obtain furthermore an explicit decomposition into irreducible representations of the homogeneous coordinate ring of the secant line variety of a Segre-Veronese variety, thus making it possible to compute the Hilbert function for this class of varieties. This can be regarded as a generalization of the computation of the degree of these secant varieties in [CS07].

Before stating the main theorem, we establish some notation. For a vector space V, V^* denotes its dual, and $\mathbb{P}V$ denotes the projective space of lines in V. If $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ is a partition, S_μ denotes the corresponding Schur functor (if $\mu_2 = 0$ we get symmetric powers, whereas if all $\mu_i = 1$, we get wedge powers). For positive integers $d_1, \cdots, d_n$, $SV_{d_1,\cdots,d_n}$ denotes the Segre-Veronese embedding of a product of n projective spaces via the complete linear system of the ample line bundle $\mathcal{O}(d_1, \cdots, d_n)$. $\sigma_2(X)$ denotes the variety of secant lines to X.

Theorem 4.1.1. *Let* $X = SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ *be a Segre-Veronese variety, where each* V_i *is a vector space of dimension at least* 2 *over a field* K *of characteristic zero. The ideal of* $\sigma_2(X)$ *is generated by* 3×3 *minors of flattenings, and moreover, for every nonnegative integer* r *we have the decomposition of the degree* r *part of its homogeneous* *coordinate ring*

$$K[\sigma_2(X)]_r = \bigoplus_{\substack{\lambda=(\lambda^1,\cdots,\lambda^n)\\ \lambda^i\vdash rd_i}} (S_{\lambda^1}V_1 \otimes \cdots \otimes S_{\lambda^n}V_n)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \max_{i=1,\cdots,n} \left\lceil \frac{\lambda_2^i}{d_i} \right\rceil, \quad e_\lambda = \lambda_2^1 + \cdots + \lambda_2^n.$$

If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

Theorem 4.1.1 has further consequences to deriving certain plethystic formulas for decomposing (in special cases) symmetric powers of triple tensor products (Corollary 4.1.3a)) and Schur functors applied to tensor products of two vector spaces (Corollary 4.1.3b)), or even symmetric pletyhsm (Corollary 4.1.4).

Finding equations for higher secant varieties of Segre-Veronese varieties turns out to be a delicate task, even in the case of two factors ($n = 2$) with not too positive embeddings (small d_1, d_2). Recent progress in this direction has been obtained by Cartwright, Erman and Oeding ([CEO10]).

Since finding precise descriptions of the equations, and more generally syzygies, of secant varieties to Segre-Veronese varieties constitutes such an intricate project, much of the current effort is directed to finding more qualitative statements. Draisma and Kuttler ([DK11]) prove that for each k, there is an uniform bound $d(k)$ such that the $(k-1)$-st secant variety of any Segre variety is cut out (set-theoretically) by equations of degree at most $d(k)$. Theorem 4.1.1 implies that $d(2) = 3$, even ideal theoretically.

For higher syzygies, Snowden ([Sno10]) proves that all the syzygies of Segre varieties are obtained from a finite amount of data via an iterative process. It would be interesting to know if the same result holds for the secant varieties. This would generalize the result of Draisma and Kuttler. For Veronese varieties, the asymptotic picture of the Betti tables is described in work of Ein and Lazarsfeld ([EL11]). Again, it would be desirable to have analogous results for secant varieties.

1.3 Hilbert Functions of Gorenstein Artin Algebras and Catalecticant Varieties

The possible Hilbert functions of graded (Artin) algebras are characterized by a classical theorem of Macaulay ([BH93, Thm. 4.2.10]), and it would be desirable to have a similar characterization for Gorenstein (Artin) algebras. The spaces of Gorenstein Artin rings with a given Hilbert function ([IK99]) are described in terms of determinantal loci of catalecticant matrices. These are matrices that generalize both generic symmetric matrices and Hankel matrices, and their ideals of minors also provide some of the equations for the secant varieties to the Veronese varieties (see Sections 2.4 and 2.5).

In [Ger99], Geramita gives a beautiful exposition of classical results about catalecticant varieties, and proposes several further questions (see also [IK99], Chapter 9). We recall the last one, which we shall answer affirmatively in Section 6.2. It is divided into two parts:

Q5a. Is it true that
$$I_3(Cat(2,d-2;n)) = I_3(Cat(t,d-t;n))$$
for all t with $2 \le t \le d-2$?

Q5b. Is it true that for $n \ge 3$ and $d \ge 4$
$$I_3(Cat(1,d-1;n)) \subsetneq I_3(Cat(2,d-2;n))?$$

Here $Cat(t,d-t;n)$ denotes the t-th generic catalecticant (see Section 2.4), and $I_3(Cat(t,d-t;n))$ is the ideal generated by its 3×3 minors. Geramita conjectures that the answers to these questions are affirmative, and furthermore, that any of the catalecticant ideals $I_3(Cat(t,d-t;n)$, $2 \le t \le d-2$, is the ideal of the secant line variety of the d-uple embedding of $\mathbb{P}^{n-1}$. Once we answer *Q5a* and *Q5b* positively, the last part of the conjecture follows from a result of Kanev ([Kan99]), which states that the ideal of the secant line variety to the Veronese variety is generated by the 3×3 minors of the first and second catalecticants (alternatively, one can apply Theorem 4.1.1 in the special case $n=1$, $d_1=d$).

The values of the Hilbert function of a graded Gorenstein Artin algebra coincide with the ranks of the catalecticants associated to its dual socle generator ([Eis95, Thm. 21.6]). Macaulay's theorem characterizing Hilbert functions of Artin algebras can thus be used to give set theoretic inclusions between ideals of minors of catalecticant matrices. This provided the motivation behind *Q5a* and *Q5b*. Geramita asks the more general question

Q4. How is Macaulay's theorem on the growth of the Hilbert function of an Artin algebra reflected in containment relations among ideals of minors of catalecticant matrices?

A partial answer to this question would be provided by the proof of the following conjecture, which is a natural generalization of *Q5a* and *Q5b* (I_k denotes the ideal of $k\times k$ minors of a matrix).

Conjecture 1.3.1. *For all $k,n \ge 2$, $d \ge 2k-2$ and t with $k-1 \le t \le d-k+1$, one has*
$$I_k(Cat(k-1,d-k+1;n)) = I_k(Cat(t,d-t;n)).$$
Moreover, the following inclusions hold:
$$I_k(Cat(1,d-1;n)) \subset I_k(Cat(2,d-2;n)) \subset \cdots \subset I_k(Cat(k-1,d-k+1;n)).$$

When $n = 2$, it is well-known (see [GP82], [Eis88], [Con98] for proofs) that

$$I_k(Cat(k-1, d-k+1; 2)) = I_k(Cat(t, d-t; 2)) \tag{1.3.1}$$

for all t with $k-1 \leq t \leq d-k+1$, and that any of these ideals is the ideal of the $(k-2)$-nd secant variety to the d-uple embedding of $\mathbb{P}^1$, hence Conjecture 1.3.1 holds in this case. For $t < k-1$ (or $t > d-k+1$), $I_k(Cat(t, d-t; 2)) = 0$ because the catalecticant matrices have less than k rows (or columns).

We prove Conjecture 1.3.1 in three special cases, namely $k = 2, 3$ and 4. The case $k = 2$ was already known, by an easy reduction to the case $n = 2$ in characteristic zero, and by [Puc98] in arbitrary characteristic. We give a short proof of Pucci's result in Section 6.1, together with a characteristic zero proof independent of (1.3.1).

Chapter 2

Preliminaries

Throughout this work, K denotes a field of characteristic 0. All the varieties we study are of finite type over K. They are usually reduced and irreducible, but we don't require this for the definition of variety (e.g. catalecticant varieties are neither reduced nor irreducible in general). $\mathbb{P}^N$ denotes the N-dimensional projective space over K. We write $\mathbb{P}W$ for $\mathbb{P}^N$ when we think of $\mathbb{P}^N$ as the space of 1-dimensional subspaces (lines) in a vector space W of dimension $N+1$ over K. Given a nonzero vector $w \in W$, we denote by $[w]$ the corresponding line. The coordinate ring of $\mathbb{P}W$ is $\mathrm{Sym}(W^*)$, the symmetric algebra on the vector space W^* of linear functionals on W.

2.1 Secant Varieties

Definition 2.1.1. Given a subvariety $X \subset \mathbb{P}^N$, the $(k-1)$-st secant variety of X, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by k points on X:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\cdots,x_k \in X} \mathbb{P}_{x_1,\cdots,x_k}}.$$

Alternatively, if we write $\mathbb{P}^N = \mathbb{P}W$ for some vector space W, and let $\hat{X} \subset W$ denote the cone over X, then we can define $\sigma_k(X)$ by specifying its cone $\widehat{\sigma_k(X)}$. This is the closure of the image of the map

$$s : \hat{X} \times \cdots \times \hat{X} \longrightarrow W,$$

defined by

$$s(x_1, \cdots, x_k) = x_1 + \cdots + x_k.$$

The main problem we are concerned with is

Problem. *Given (the equations of) X, determine (the equations of) $\sigma_k(X)$.*

More precisely, given the homogeneous ideal $I(X)$ of the subvariety $X \subset \mathbb{P}W$, we would like to describe the generators of $I(\sigma_k(X))$. Alternatively, we would like to understand the homogeneous coordinate ring of $\sigma_k(X)$, which we denote by $K[\sigma_k(X)]$. As we will see, this is a difficult problem even in the case when X is simple, i.e. isomorphic to a projective space (or a product of such). There is thus little hope of giving an uniform satisfactory answer in the generality we posed the problem. However, the following observation provides a general approach to the problem, which we exploit in the future chapters.

The ideal/homogeneous coordinate ring of a subvariety $Y \subset \mathbb{P}W$ coincides with the ideal/affine coordinate ring of its cone $\hat{Y} \subset W$, hence our problem is equivalent to understanding $I(\widehat{\sigma_k(X)})$ and $K[\widehat{\sigma_k(X)}]$. The morphism s of affine varieties defined above corresponds to a ring map

$$s^{\#} : \mathrm{Sym}(W^*) \to K[\hat{X} \times \cdots \times \hat{X}] = K[\hat{X}] \otimes \cdots \otimes K[\hat{X}].$$

We have that $I(\widehat{\sigma_k(X)})$ and $K[\widehat{\sigma_k(X)}]$ are the kernel and image respectively of $s^{\#}$. The main focus for us will be on the case when X is a Segre-Veronese variety (described in the following section), and $k = 2$.

2.2 Segre-Veronese Varieties

Consider vector spaces $V_1, \cdots, V_n$ of dimensions $m_1, \cdots, m_n \geq 2$ respectively, with duals $V_1^*, \cdots, V_n^*$, and positive integers $d_1, \cdots, d_n$. We let

$$X = \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \cdots, d_n)$. Explicitly, X is the image of the map

$$SV_{d_1,\cdots,d_n} : \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^* \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*)$$

given by

$$([e_1], \cdots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

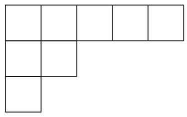
We call X a Segre-Veronese variety.

For such X we prove that $I(\sigma_2(X))$ is generated in degree 3 and we describe the decomposition of $K[\sigma_2(X)]$ into a sum of irreducible representations of the product of general linear groups $GL(V_1) \times \cdots \times GL(V_n)$ (Theorem 4.1.1).

When $n = 1$ we set $d = d_1$, $V = V_1$. The image of SV_d is the d-th Veronese embedding, or d-uple embedding of the projective space $\mathbb{P}V^*$, which we denote by $\mathrm{Ver}_d(\mathbb{P}V^*)$. When $d_1 = \cdots = d_n = 1$, the image of $SV_{1,1,\cdots,1}$ is the Segre variety $\mathrm{Seg}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$. An element of $\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*$ is called a (partially symmetric) tensor. The points in the cone over the Segre-Veronese variety are called pure tensors.

2.3 Representation Theory

We refer the reader to [FH91] for the basic representation theory of symmetric and general linear groups. Given a positive integer r, a partition μ of r is a nonincreasing sequence of nonnegative integers $\mu_1 \geq \mu_2 \geq \cdots$ with $r = \sum \mu_i$. We write $\mu = (\mu_1, \mu_2, \cdots)$. Alternatively, if μ is a partition having i_j parts equal to μ_j for all j, then we write $\mu = (\mu_1^{i_1} \mu_2^{i_2} \cdots)$. To a partition $\mu = (\mu_1, \mu_2, \cdots)$ we associate a Young diagram which consists of left-justified rows of boxes, with μ_i boxes in the i-th row. For $\mu = (5, 2, 1)$, the corresponding Young diagram is



For a vector space W, a positive integer r and a partition μ of r, we denote by $S_\mu W$ the corresponding irreducible representation of $GL(W)$: S_μ are commonly known as Schur functors, and we make the convention that $S_{(d)}$ denotes the symmetric power functor, while $S_{(1^d)}$ denotes the exterior power functor. We write S_r for the symmetric group on r letters, and $[\mu]$ for the irreducible S_r-representation corresponding to μ: $[(d)]$ denotes the trivial representation and $[(1^d)]$ denotes the sign representation.

Given a positive integer n and a sequence of nonnegative integers $\underline{r} = (r_1, \cdots, r_n)$, we define an n-partition of $\underline{r}$ to be an n-tuple of partitions $\lambda = (\lambda^1, \cdots, \lambda^n)$, with λ^j partition of r_j, $j = 1, \cdots, n$. We write $\lambda^j \vdash r_j$ and $\lambda \vdash^n \underline{r}$. Given vector spaces $V_1, \cdots, V_n$ as above, we often write $GL(V)$ for $GL(V_1) \times \cdots \times GL(V_n)$. We write $S_\lambda V$ for the irreducible $GL(V)$-representation $S_{\lambda^1} V_1 \otimes \cdots \otimes S_{\lambda^n} V_n$. Similarly, we write $[\lambda]$ for the irreducible representation $[\lambda^1] \otimes \cdots \otimes [\lambda^n]$ of the n-fold product of symmetric groups $S_{\underline{r}} = S_{r_1} \times \cdots \times S_{r_n}$. We have

Lemma 2.3.1 (Schur-Weyl duality)**.**

$$V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n} = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda] \otimes S_\lambda V.$$

Most of the group actions we consider are left actions, denoted by $\cdot$. We use the symbol $*$ for right actions, to distinguish them from left actions.

For a subgroup $H \subset G$ and representations U of H and W of G, we write

$$\mathrm{Ind}_H^G(U) = K[G] \otimes_{K[H]} U, \text{ and } \mathrm{Res}_H^G(W) = W_H,$$

for the induced representation of U and restricted representation of W, where $K[M]$ denotes the group algebra of a group M, and W_H is just W, regarded as an H-module. We write W^G for the G-invariants of the representation W, i.e.

$$W^G = \mathrm{Hom}_G(\mathbf{1}, W) \subset \mathrm{Hom}_K(\mathbf{1}, W) = W,$$

where $\mathbf{1}$ denotes the trivial representation of G.

Remark 2.3.2. If G is finite, let

$$s_G = \sum_{g \in G} g \in K[G].$$

We can realize W^G as the image of the map $W \longrightarrow W$ given by

$$w \mapsto s_G \cdot w.$$

Assume furthermore that $H \subset G$ is a subgroup, and let s_H denote the corresponding element in $K[H]$. We have a natural inclusion of the trivial representation of H

$$\mathbf{1} \hookrightarrow K[H], \quad 1 \mapsto s_H,$$

which after tensoring with $K[G]$ becomes

$$\mathrm{Ind}_H^G(\mathbf{1}) = K[G] \otimes_{K[H]} \mathbf{1} \hookrightarrow K[G] \otimes_{K[H]} K[H] \simeq K[G],$$

so that we can identify $\mathrm{Ind}_H^G(\mathbf{1})$ with $K[G] \cdot s_H$.

We have

Lemma 2.3.3 (Frobenius reciprocity).

$$W^H = \mathrm{Hom}_H(\mathbf{1}, \mathrm{Res}_H^G(W)) = \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbf{1}), W).$$

Given an n-partition $\lambda = (\lambda^1, \cdots, \lambda^n)$ of $\underline{r}$, we define an n-tableau of shape λ to be an n-tuple $T = (T^1, \cdots, T^n)$, which we usually write as $T^1 \otimes \cdots \otimes T^n$, where each T^i is a tableau of shape λ^i. A tableau is canonical if its entries index its boxes consecutively from left to right, and top to bottom. We say that T is canonical if each T^i is, in which case we write T_λ for T. If $T = (\lambda^1, \lambda^2)$, with $\lambda^1 = (3, 2)$, $\lambda^2 = (3, 1, 1)$, then the canonical 2-tableau of shape λ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}.$$

We consider the subgroups of $S_{\underline{r}}$

$$R_\lambda = \{g \in S_{\underline{r}} : g \ \text{ preserves each row of } \ T_\lambda\},$$

$$C_\lambda = \{g \in S_{\underline{r}} : g \ \text{ preserves each column of } \ T_\lambda\}$$

and define the symmetrizers

$$a_\lambda = \sum_{g \in R_\lambda} g, \quad b_\lambda = \sum_{g \in C_\lambda} \mathrm{sgn}(g) \cdot g, \quad c_\lambda = a_\lambda \cdot b_\lambda,$$

with $\operatorname{sgn}(g) = \prod_i \operatorname{sgn}(g_i)$ for $g = (g_1, \cdots, g_n) \in S_{\underline{r}}$, where $\operatorname{sgn}(g_i)$ denotes the signature of the permutation g_i.

The $GL(V)$- (or $S_{\underline{r}}$-) representations W that we consider decompose as a direct sum of $S_\lambda V$'s (or $[\lambda]$'s) with $\lambda \vdash^n \underline{r}$. We write

$$W = \bigoplus_\lambda W_\lambda,$$

where $W_\lambda \simeq (S_\lambda V)^{m_\lambda}$ (or $W_\lambda \simeq [\lambda]^{m_\lambda}$) for some nonnegative integer $m_\lambda = m_\lambda(W)$, called the multiplicity of $S_\lambda V$ (or $[\lambda]$) in W. We call W_λ the λ-part of the representation W.

Recall that m_j denotes the dimension of V_j, $j = 1, \cdots, n$. We fix bases

$$\mathcal{B}_j = \{x_{ij} : i = 1, \cdots, m_j\}$$

for V_j ordered by $x_{ij} > x_{i+1,j}$. We choose the maximal torus $T = T_1 \times \cdots \times T_n \subset GL(V)$, with T_j being the set of diagonal matrices with respect to $\mathcal{B}_j$. We choose the Borel subgroup of $GL(V)$ to be $B = B_1 \times \cdots \times B_n$, where B_j is the subgroup of upper triangular matrices in $GL(V_j)$ with respect to $\mathcal{B}_j$. Given a $GL(V)$-representation W, a weight vector w with weight $a = (a_1, \cdots, a_n)$, $a_i \in T_i^*$, is a nonzero vector in W with the property that for any $t = (t_1, \cdots, t_n) \in T$,

$$t \cdot w = a_1(t_1) \cdots a_n(t_n) w.$$

The vectors with this property form a vector space called the a-weight space of W, which we denote by $\mathrm{wt}_a(W)$.

A highest weight vector of a $GL(V)$-representation W is an element $w \in W$ invariant under B. $W = S_\lambda V$ has a unique (up to scaling) highest weight vector w with corresponding weight $\lambda = (\lambda^1, \cdots, \lambda^n)$. In general, we define the λ-highest weight space of a $GL(V)$-representation W to be the set of highest weight vectors in W with weight λ, and denote it by $\mathrm{hwt}_\lambda(W)$. If W is an $S_{\underline{r}}$-representation, the λ-highest weight space of W is the vector space $\mathrm{hwt}_\lambda(W) = c_\lambda \cdot W \subset W$, where c_λ is the Young symmetrizer defined above. In both cases, $\mathrm{hwt}_\lambda(W)$ is a vector space of dimension $m_\lambda(W)$.

2.4 Catalecticant varieties

Given a vector space V of dimension n over K, with basis $\mathcal{B} = \{x_1, \cdots, x_n\}$, we consider its dual space V^* with dual basis $\mathcal{E} = \{e_1, \cdots, e_n\}$. For every positive integer d we get a basis of $S_{(d)}V^*$ consisting of divided power monomials $e^{(\alpha)}$ of degree d in the e_i's, as follows. If $\alpha \subset \{1, \cdots, n\}$ is a multiset of size $|\alpha| = d$, then we write e^α for the monomial

$$\prod_{i \in \alpha} e_i.$$

We often identify α with the multiindex $(\alpha_1, \cdots, \alpha_n)$, where α_i represents the number of occurrences of i in the multiset α. We write $e^{(\alpha)}$ for $e^\alpha/\alpha!$, where $\alpha! = \alpha_1! \cdots \alpha_n!$. For $a, b > 0$ with $a + b = d$ we get a divided power multiplication map $S_{(a)}V^* \otimes S_{(b)}V^* \to S_{(d)}V^*$, sending $e^{(\alpha)} \otimes e^{(\beta)}$ to $e^{(\alpha\cup\beta)}$. We can represent this via a multiplication table whose rows and columns are indexed by multisets of sizes a and b respectively, and whose entry in the (α, β) position is $e^{(\alpha\cup\beta)}$. The generic catalecticant matrix $Cat(a, b; n)$ is defined to be the matrix obtained from this multiplication table by replacing each $e^{(\alpha\cup\beta)}$ with the variable $z_{\alpha\cup\beta}$, where $(z_\gamma)_{|\gamma|=d} \subset S_{(d)}V$ is the dual basis to $(e^{(\gamma)})_{|\gamma|=d} \subset S_{(d)}V^*$.

One can also think of z_γ's as the coefficients of the generic form of degree d in the e_i's, $F = \sum z_\gamma e^{(\gamma)}$. Specializing the z_γ's we get an actual form $f \in S_{(d)}V^*$, and we denote the corresponding catalecticant matrix by $Cat_f(a, b; n)$. Any such form f is the dual socle generator of some Gorenstein Artin algebra A ([Eis95, Thm. 21.6]) with socle degree d and Hilbert function

$$h_i(A) = \operatorname{rank}(Cat_f(i, d-i; n)).$$

Macaulay's theorem on the growth of the Hilbert function of an Artin algebra ([BH93, Thm. 4.2.10]) implies that if $h_i < k$ for some $i \geq k-1$, then the function becomes nonincreasing from that point on. In particular, since A is Gorenstein, h is symmetric, so if $h_i < k$ for some $k - 1 \leq i \leq d - k + 1$, then we have

$$h_1 \leq h_2 \leq \cdots \leq h_{k-1} = h_k = \cdots = h_{d-k+1} \geq h_{d-k+2} \geq \cdots \geq h_d.$$

If we denote by $I_k(i) = I_k(Cat(i, d-i; n))$ the ideal of $k \times k$ minors of the i-th generic catalecticant, then the remarks above show that whenever $k - 1 \leq d - k + 1$ we have the following up-to-radical relations:

$$I_k(1) \subset \cdots \subset I_k(k-1) = \cdots = I_k(d-k+1) \supset \cdots \supset I_k(d-1).$$

Conjecture 1.3.1 states that these relations hold exactly. We prove the conjecture in the cases $k = 2, 3$ and 4 in Chapter 6.

2.5 Flattenings

Given decompositions $d_i = a_i + b_i$, with $a_i, b_i \geq 0$, $i = 1, \cdots, n$, we let $A = (a_1, \cdots, a_n)$, $B = (b_1, \cdots, b_n)$, so that $\underline{d} = (d_1, \cdots, d_n) = A + B$, and embed

$$\operatorname{Sym}^{d_1} V_1^* \otimes \cdots \otimes \operatorname{Sym}^{d_n} V_n^* \hookrightarrow V_A^* \otimes V_B^*$$

in the usual way, where

$$V_A = \operatorname{Sym}^{a_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{a_n} V_n, \quad V_B = \operatorname{Sym}^{b_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{b_n} V_n.$$

This embedding allows us to flatten any tensor in $\operatorname{Sym}^{d_1} V_1^* \otimes \cdots \otimes \operatorname{Sym}^{d_n} V_n^*$ to a 2-tensor, i.e. a matrix, in $V_A^* \otimes V_B^*$. We call such a matrix an (A, B)-flattening of our tensor. If

$|A| = a_1 + ... + a_n$ then we also say that this matrix is an $|A|$-flattening, or a $|B|$-flattening, by symmetry. In the case of one factor ($n = 1$), the flattening matrices are precisely the catalecticant matrices described in the previous section.

We obtain an inclusion

$$SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*) \hookrightarrow \mathrm{Seg}(\mathbb{P}V_A^* \times \mathbb{P}V_B^*),$$

and consequently

$$\sigma_k(SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)) \hookrightarrow \sigma_k(\mathrm{Seg}(\mathbb{P}V_A^* \times \mathbb{P}V_B^*)),$$

where the latter secant variety coincides with (the projectivization of) the set of matrices of rank at most k in $V_A^* \otimes V_B^*$. This set is cut out by the $(k+1) \times (k+1)$ minors of the generic matrix in $V_A^* \otimes V_B^*$. This observation yields equations for the secant varieties of Segre-Veronese varieties (see also [Lan]).

Lemma 2.5.1. *For any decomposition $\underline{d} = A + B$ and any $k \geq 1$, the ideal of $(k+1) \times (k+1)$ minors of the generic matrix given by the (A, B)-flattening of $\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*$ is contained in the ideal of $\sigma_k(SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$.*

Definition 2.5.2. We write $F_{A,B}^{k+1,r}(V) = F_{A,B}^{k+1,r}(V_1, \cdots, V_n)$ for the degree r part of the ideal of $(k+1) \times (k+1)$ minors of the (A, B)-flattening.

Note that the invariant way of writing the generators of the ideal of $(k+1) \times (k+1)$ minors of the (A, B)-flattening in the preceding lemma ($F_{A,B}^{k+1,k+1}(V)$) is as the image of the composition

$$\bigwedge^{k+1} V_A \otimes \bigwedge^{k+1} V_B \hookrightarrow \mathrm{Sym}^{k+1}(V_A \otimes V_B) \longrightarrow \mathrm{Sym}^{k+1}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n),$$

where the first map is the usual inclusion map, while the last one is induced by the multiplication maps

$$\mathrm{Sym}^{a_i} V_i \otimes \mathrm{Sym}^{b_i} V_i \longrightarrow \mathrm{Sym}^{d_i} V_i.$$

2.6 The ideal and coordinate ring of a Segre-Veronese variety

If $X = SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$, then the ideal $I(X)$ is generated by 2×2 minors of flattenings, i.e. when $k = 1$ the equations described in Lemma 2.5.1 are sufficient to generate the ideal of the corresponding variety. As for the homogeneous coordinate ring of a Segre-Veronese variety, we have the decomposition

$$K[X] = \bigoplus_{r \geq 0} (\mathrm{Sym}^{rd_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{rd_n} V_n). \tag{*}$$

This decomposition will turn out to be useful in the next chapter, in conjunction with the map $s^{\#}$ defined in Section 2.1. In Chapter 4 we give a description of $K[\sigma_2(X)]$ analogous to (*), and prove that the 3×3 minors of flattenings generate the homogeneous ideal of $\sigma_2(X)$.

The statements above regarding the ideal and coordinate ring of a Segre-Veronese variety hold more generally for rational homogeneous varieties (G/P), and have been obtained in unpublished work by Kostant (see [Lan]).

Chapter 3

Equations of the secant varieties of a Segre-Veronese variety

This chapter introduces the main new tool for understanding the equations and coordinate rings of the secant varieties of Segre-Veronese varieties, from a representation theoretic/combinatorial perspective. All the subsequent chapters are based on the ideas described here. The usual method for analyzing the secant varieties of Segre-Veronese varieties is based on the representation theory of general linear groups. We review some of its basic ideas, including the notion of inheritance, in Section 3.1. The new insight of restricting the analysis to special equations of the secant varieties, the "generic equations", is presented in Section 3.2. More precisely, we use Schur-Weyl duality to translate questions about representations of general linear groups into questions about representations of symmetric groups and tableaux combinatorics. The relationship between the two situations is made precise in Section 3.3. One should think of the "generic equations" as a set of equations that give rise by specialization to all the equations of the secant varieties of Segre-Veronese varieties. Similarly, we have the "generic flattenings" which by specialization yield the usual flattenings.

3.1 Multi-prolongations and inheritance

In this section $V_1, \cdots, V_n$ are (as always) vector spaces over a field K of characteristic zero. We switch from the Sym^d notation to the more compact Schur functor notation $S_{(d)}$ described in Section 2.3. The homogeneous coordinate ring of $\mathbb{P}(S_{(d_1)}V_1^* \otimes \cdots \otimes S_{(d_n)}V_n^*)$ is

$$S = \mathrm{Sym}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n),$$

the symmetric algebra of the vector space $S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n$. This vector space has a natural basis $\mathcal{B} = \mathcal{B}_{d_1,\cdots,d_n}$ consisting of tensor products of monomials in the elements of the bases $\mathcal{B}_1, \cdots, \mathcal{B}_n$ of $V_1, \cdots, V_n$. We write this basis, suggestively, as

$$\mathcal{B} = \mathrm{Sym}^{d_1} \mathcal{B}_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} \mathcal{B}_n.$$

We can index the elements of $\mathcal{B}$ by n-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$ of multisets α_i of size d_i with entries in $\{1, \cdots, m_i = \dim(V_i)\}$, as follows. The α-th element of the basis $\mathcal{B}$ is

$$z_\alpha = (\prod_{i_1 \in \alpha_1} x_{i_1,1}) \otimes \cdots \otimes (\prod_{i_n \in \alpha_n} x_{i_n,n}),$$

and we think of z_α as a linear form in S.

We can therefore identify S with the polynomial ring $K[z_\alpha]$. One would like to have a precise description of the ideal $I \subset S$ of polynomials vanishing on $\sigma_k(SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$, but this is a very difficult problem, as mentioned in the introduction. We obtain such a description for the case $k = 2$ in Theorem 4.1.1. The case $k = 1$ was already known, as described in Section 2.6.

Given a positive integer r and a partition $\mu = (\mu_1, \cdots, \mu_t) \vdash r$, we consider the set $\mathcal{P}_\mu$ of all (unordered) partitions of $\{1, \cdots, r\}$ of shape μ, i.e.

$$\mathcal{P}_\mu = \left\{ A = \{A_1, \cdots, A_t\} : |A_i| = \mu_i \text{ and } \bigsqcup_{i=1}^{t} A_i = \{1, \cdots, r\} \right\},$$

as opposed to the set of ordered partitions where we take instead $A = (A_1, \cdots, A_t)$.

Definition 3.1.1. For a partition $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ of r, we consider the map

$$\pi_\mu : S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n) \longrightarrow \bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n),$$

given by

$$z_1 \cdots z_r \mapsto \sum_{A \in \mathcal{P}_\mu} \bigotimes_{j=1}^{s} \prod_{\substack{B \in A \\ |B| = \mu_j}} m(z_i : i \in B),$$

where

$$m : (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_j} \longrightarrow S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n$$

denotes the usual componentwise multiplication map.

We write $\pi_\mu(V)$ or $\pi_\mu(V_1, \cdots, V_n)$ for the map π_μ just defined, when we want to distinguish it from its generic version (Definition 3.2.5). We also write $U_r^{\underline{d}}(V) = U_r^{\underline{d}}(V_1, \cdots, V_n)$ and $U_\mu^{\underline{d}}(V) = U_\mu^{\underline{d}}(V_1, \cdots, V_n)$ for the source and target of $\pi_\mu(V)$ respectively (see Definitions 3.2.1 and 3.2.4 for the generic versions of these spaces).

A more invariant way of stating Definition 3.1.1 is as follows. If $\mu = (\mu_1, \cdots, \mu_t)$, then the map π_μ is the composition between the usual inclusion

$$S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n) \hookrightarrow (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes r} =$$

$$(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_1} \otimes \cdots \otimes (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_t}.$$

and the tensor product of the natural multiplication maps

$$m : (S_{(d_i)}V_i)^{\otimes \mu_j} \longrightarrow S_{(\mu_j d_i)}V_i.$$

Example 3.1.2. Let $n = 2$, $d_1 = 2$, $d_2 = 1$, $r = 4$, $\mu = (2,2) = (2^2)$, $\dim(V_1) = 2$, $\dim(V_2) = 3$. Take

$$z_1 = z_{(\{1,2\},\{1\})}, \quad z_2 = z_{(\{1,1\},\{3\})}, \quad z_3 = z_{(\{1,1\},\{1\})}, \quad z_4 = z_{(\{2,2\},\{2\})}.$$

We have

$$\pi_\mu(z_1 \cdot z_2 \cdot z_3 \cdot z_4) = m(z_1, z_2) \cdot m(z_3, z_4) + m(z_1, z_3) \cdot m(z_2, z_4) + m(z_1, z_4) \cdot m(z_2, z_3) =$$

$$z_{(\{1,1,1,2\},\{1,3\})} \cdot z_{(\{1,1,2,2\},\{1,2\})} + z_{(\{1,1,1,2\},\{1,1\})} \cdot z_{(\{1,1,2,2\},\{2,3\})} + z_{(\{1,2,2,2\},\{1,2\})} \cdot z_{(\{1,1,1,1\},\{1,3\})}.$$

A more "visual" way of representing the monomials in $\mathrm{Sym}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n) = K[z_\alpha]$ and the maps π_μ is as follows. We identify each z_α with an $1 \times n$ block with entries the multisets α_i:

$$z_\alpha = \begin{array}{|c|c|c|c|}\hline \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \hline \end{array}.$$

We represent a monomial $m = z_{\alpha^1} \cdots z_{\alpha^r}$ of degree r as an $r \times n$ block M, whose rows correspond to the variables z_{α^i} in the way described above.

$$m \equiv M = \begin{array}{|c|c|c|c|}\hline \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_n^1 \\ \hline \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \alpha_1^r & \alpha_2^r & \cdots & \alpha_n^r \\ \hline \end{array}$$

Note that the order of the rows is irrelevant, since the z_{α^i}'s commute. The way π_μ acts on an $r \times n$ block M is as follows: it partitions in all possible ways the set of rows of M into subsets of sizes equal to the parts of μ, collapses the elements of each subset into a single row, and takes the sum of all blocks obtained in this way. Here by collapsing a set of rows we mean taking the columnwise union of the entries of the rows. More precisely, if M is the $r \times n$ block corresponding to $z_{\alpha^1} \cdots z_{\alpha^r}$ and $\mu = (\mu_1, \cdots, \mu_t)$, then

$$\pi_\mu M = \sum_{\substack{A \in \mathcal{P}_\mu \\ A = \{A_1, \cdots, A_t\}}} \begin{array}{|c|c|c|}\hline \cdots & \bigcup_{i \in A_1} \alpha_k^i & \cdots \\ \hline \cdots & \bigcup_{i \in A_2} \alpha_k^i & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline \cdots & \bigcup_{i \in A_t} \alpha_k^i & \cdots \\ \hline \end{array}.$$

Note that if two A_i's have the same cardinality, then the variables corresponding to their rows commute, so we can harmlessly interchange them.

Example 3.1.3. With these conventions, we can rewrite Example 3.1.2 as

$$\begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 1,1 & 3 \\ \hline 1,1 & 1 \\ \hline 2,2 & 2 \\ \hline \end{array} \xrightarrow{\pi_{(2,2)}} \begin{array}{|c|c|}\hline 1,1,1,2 & 1,3 \\ \hline 1,1,2,2 & 1,2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,1,1,2 & 1,1 \\ \hline 1,1,2,2 & 2,3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,2,2,2 & 1,2 \\ \hline 1,1,1,1 & 1,3 \\ \hline \end{array}.$$

Proposition 3.1.4 (Multi-prolongations, [Lan]). *For a positive integer r, the polynomials of degree r vanishing on $\sigma_k(SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$ are precisely the elements of $S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)$ in the intersection of the kernels of the maps π_μ, where μ ranges over all partitions of r with (at most) k parts.*

Proof. Let X denote the Segre-Veronese variety $SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$. As in Section 2.1, there exists a map ($s^\#$, which we now denote π)

$$\pi : \mathrm{Sym}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n) \longrightarrow K[X]^{\otimes k},$$

whose kernel and image coincide with the ideal and homogeneous coordinate ring respectively, of $\sigma_k(X)$. Using the description of $K[X]$ given in Section 2.6, we obtain that the degree r part of the target of π is

$$(K[X]^{\otimes k})_r = \bigoplus_{\mu_1+\cdots+\mu_k=r} (S_{(\mu_1 d_1)}V_1 \otimes \cdots \otimes S_{(\mu_1 d_n)}V_n) \otimes \cdots \otimes (S_{(\mu_k d_1)}V_1 \otimes \cdots \otimes S_{(\mu_k d_n)}V_n).$$

The degree r component of π, which we call π_r, is then a direct sum of maps π_μ as in Definition 3.1.1, where μ ranges over partitions of r with at most k parts. The conclusion of the proposition now follows. To see that it's enough to only consider partitions with exactly k parts, note that if μ has fewer than k parts, and $\widehat{\mu}$ is a partition obtained by subdividing μ (splitting some of the parts of μ into smaller pieces), then π_μ factors through (up to a multiplicative factor) $\pi_{\widehat{\mu}}$, hence $\ker(\pi_\mu) \supset \ker(\pi_{\widehat{\mu}})$, so the contribution of $\ker(\pi_\mu)$ to the intersection of kernels is superfluous. $\square$

Definition 3.1.5 (Multi-prolongations). We write $I_\mu(V) = I_\mu^{\underline{d}}(V)$ for the kernel of the map $\pi_\mu(V)$, and $I_r(V) = I_r^{\underline{d}}(V)$ for the intersection of the kernels of the maps $\pi_\mu(V)$ as μ ranges over partitions of r with k parts. I.e. $I_r(V)$ is the degree r part of the ideal of $\sigma_k(SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$.

Given the description of the ideal of $\sigma_k(X)$ as the kernel of the $GL(V)$-equivariant map π, we now proceed to analyze π irreducible representation by irreducible representation. That is, we fix a positive integer r and an n-partition $\lambda = (\lambda^1, \cdots, \lambda^n)$ of $(rd_1, \cdots, rd_n)$, and we restrict π to the λ-parts of its source and target. The map π depends functorially on the vector spaces $V_1, \cdots, V_n$, and its kernel and image stabilize from a representation theoretic point of view as the dimensions of the V_i's increase. More precisely, we have the following

Proposition 3.1.6 (Inheritance, [Lan]). *Fix an n-partition $\lambda \vdash^n r \cdot (d_1, \cdots, d_n)$. Let l_j denote the number of parts of λ^j, for $j = 1, \cdots, n$. Then the multiplicities of $S_\lambda V$ in the kernel and image respectively of π are independent of the dimensions m_j of the V_j's, as long as $m_j \geq l_j$. Moreover, if some l_j is larger than k, then $S_\lambda V$ doesn't occur as a representation in the image of π.*

Proof. The last statement follows from the representation theoretic description of the coordinate ring of a Segre-Veronese variety, and Pieri's rule: every irreducible representation $S_\lambda V$ occurring in $K[X]^{\otimes k}$ must have the property that each λ^j has at most k parts.

As for the first part, note that π is completely determined by what it does on the λ-highest weight vectors, and that the λ-highest weight vector of an irreducible representation $S_\lambda V$ only depends on the first l_j elements of the basis $\mathcal{B}_j$, for $j = 1, \cdots, n$. $\square$

We just saw in the previous proposition that (the λ-part of) π is essentially insensitive to expanding or shrinking the vector spaces V_i, as long as their dimensions remain larger than l_i. Also, the last part of the proposition allows us to concentrate on n-partitions λ where each λ^i has at most k parts. To understand π, we thus have the freedom to pick the dimensions of the V_i's to be positive integers at least equal to k. It might seem natural then to pick these dimensions as small as possible (equal to k), and understand the kernel and image of π in that situation. However, we choose not to do so, and instead we fix a positive degree r and concentrate our attention on the map π_r, the degree r part of π. We assume that

$$\dim(V_i) = r \cdot d_i, \quad i = 1, \cdots, n.$$

The reason for this assumption is that now the $\mathfrak{sl}$ zero-weight spaces of the source and target of π_r are nonempty and generate the corresponding representations. Therefore π_r is determined by its restriction to these zero-weight spaces, which suddenly makes our problem combinatorial: the zero-weight spaces are modules over the Weyl group, which is just the product of symmetric groups $S_{rd_1} \times \cdots \times S_{rd_n}$, allowing us to use the representation theory of the symmetric groups to analyze the map π_r. We call this reduction the "generic case", because the $\mathfrak{sl}$ zero-weight subspace of $S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)$ is the subspace containing the most generic tensors.

3.2 The "generic case"

3.2.1 Generic multi-prolongations

We let $\underline{d}, \underline{r}$ denote the sequences of numbers $(d_1, \cdots, d_n)$ and $r \cdot \underline{d} = (rd_1, \cdots, rd_n)$ respectively. We let $S_{\underline{r}}$ denote the product of symmetric groups $S_{rd_1} \times \cdots \times S_{rd_n}$, the Weyl group of the Lie algebra $\mathfrak{sl}(V)$ of $GL(V)$ (recall that $\dim(V_j) = m_j = rd_j$ for $j = 1, \cdots, n$).

Definition 3.2.1. We denote by $U_r^{\underline{d}}$ the $\mathfrak{sl}(V)$ zero-weight space of the representation $S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)$. $U_r^{\underline{d}}$ has a basis consisting of monomials

$$m = z_{\alpha^1} \cdots z_{\alpha^r},$$

where for each j, the elements of $\{\alpha_j^1, \cdots, \alpha_j^r\}$ form a partition of the set $\{1, \cdots, rd_j\}$, with $|\alpha_j^i| = d_j$. Alternatively, $U_r^{\underline{d}}$ has a basis consisting of $r \times n$ blocks M, where each column of M yields a partition of the set $\{1, \cdots, rd_j\}$ with r equal parts.

Example 3.2.2. For $n = 2$, $d_1 = 2$, $d_2 = 1$, $r = 4$, a typical element of $U_r^{\underline{d}}$ is

$$M = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 2,3 & 4 \\ \hline 4,5 & 2 \\ \hline 7,8 & 3 \\ \hline \end{array} = z_{(\{1,6\},\{1\})} \cdot z_{(\{2,3\},\{4\})} \cdot z_{(\{4,5\},\{2\})} \cdot z_{(\{7,8\},\{3\})} = m.$$

$S_{\underline{r}}$ acts on $U_r^{\underline{d}}$ by letting its j-th factor S_{rd_j} act on the j-th columns of the blocks M described above. As an abstract representation, we have

$$U_r^{\underline{d}} \simeq \mathrm{Ind}_{(S_{d_1} \times \cdots \times S_{d_n})^r \wr S_r}^{S_{\underline{r}}}(\mathbf{1}),$$

where $\wr$ denotes the wreath product of $(S_{d_1} \times \cdots \times S_{d_n})^r$ with S_r, and $\mathbf{1}$ denotes the trivial representation (we will say more about this in the following section). The dimension of the space $U_r^{\underline{d}}$ is

$$N = N_r^{\underline{d}} = \frac{(rd_1)!(rd_2)! \cdots (rd_n)!}{(d_1!d_2! \cdots d_n!)^r \cdot r!}.$$

Example 3.2.3. Continuing Example 3.2.2, let $\sigma = (\sigma_1, \sigma_2) \in S_8 \times S_4$, with $\sigma_1 = (1,2)(5,3,7)$, $\sigma_2 = (1,4,3)$, in cycle notation. Then

$$\sigma \cdot M = \begin{array}{|c|c|}\hline 2,6 & 4 \\ \hline 1,7 & 3 \\ \hline 4,3 & 2 \\ \hline 5,8 & 1 \\ \hline \end{array},$$

or

$$\sigma \cdot m = z_{(\{2,6\},\{4\})} \cdot z_{(\{1,7\},\{3\})} \cdot z_{(\{4,3\},\{2\})} \cdot z_{(\{5,8\},\{1\})}.$$

Definition 3.2.4. For a partition μ written in multiplicative notation $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ as in Definition 3.1.1, we define the space $U_\mu^{\underline{d}}$ to be the $\mathfrak{sl}$ zero-weight space of the representation

$$\bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n).$$

Writing $\mu = (\mu_1, \cdots, \mu_t)$ we can realize $U^{\underline{d}}_{\mu}$ as the vector space with a basis consisting of $t \times n$ blocks M with the entry in row i and column j consisting of $\mu_i \cdot d_j$ elements from the set $\{1, \cdots, rd_j\}$, in such a way that each column of M represents a partition of $\{1, \cdots, rd_j\}$. As usual, we identify two blocks if they differ by permutations of rows of the same size, i.e. corresponding to equal parts of μ. Note that when $\mu = (1^r)$ we get $U^{\underline{d}}_{\mu} = U^{\underline{d}}_{r}$, recovering Definition 3.2.1.

We can now define the generic version of the map π_μ from Definition 3.1.1:

Definition 3.2.5. For a partition $\mu \vdash r$ as in Definition 3.1.1, we define the map

$$\pi_\mu : U^{\underline{d}}_{r} \longrightarrow U^{\underline{d}}_{\mu},$$

to be the restriction of the map from Definition 3.1.1 to the $\mathfrak{sl}$ zero-weight spaces of the source and target.

Example 3.2.6. The generic analogue of Example 3.1.3 is:

$$\begin{array}{|c|c|}\hline 1,6 & 1\\ \hline 2,3 & 4\\ \hline 4,5 & 2\\ \hline 7,8 & 3\\ \hline\end{array} \xrightarrow{\pi_{(2,2)}} \begin{array}{|c|c|}\hline 1,2,3,6 & 1,4\\ \hline 4,5,7,8 & 2,3\\ \hline\end{array} + \begin{array}{|c|c|}\hline 1,4,5,6 & 1,2\\ \hline 2,3,7,8 & 3,4\\ \hline\end{array} + \begin{array}{|c|c|}\hline 1,6,7,8 & 1,3\\ \hline 2,3,4,5 & 2,4\\ \hline\end{array}.$$

If instead of the partition $(2,2)$ we take $\mu = (2,1,1) = (1^2 2)$, then we have

$$\begin{array}{|c|c|}\hline 1,6 & 1\\ \hline 2,3 & 4\\ \hline 4,5 & 2\\ \hline 7,8 & 3\\ \hline\end{array} \xrightarrow{\pi_{(2,1,1)}} \begin{array}{|c|c|}\hline 1,2,3,6 & 1,4\\ \hline 4,5 & 2\\ \hline 7,8 & 3\\ \hline\end{array} + \begin{array}{|c|c|}\hline 1,4,5,6 & 1,2\\ \hline 2,3 & 4\\ \hline 7,8 & 3\\ \hline\end{array} +$$

$$\begin{array}{|c|c|}\hline 1,6,7,8 & 1,3\\ \hline 2,3 & 4\\ \hline 4,5 & 2\\ \hline\end{array} + \begin{array}{|c|c|}\hline 2,3,4,5 & 2,4\\ \hline 1,6 & 1\\ \hline 7,8 & 3\\ \hline\end{array} + \begin{array}{|c|c|}\hline 2,3,7,8 & 3,4\\ \hline 1,6 & 1\\ \hline 4,5 & 2\\ \hline\end{array} + \begin{array}{|c|c|}\hline 4,5,7,8 & 2,3\\ \hline 1,6 & 1\\ \hline 2,3 & 4\\ \hline\end{array}.$$

Note that if we compose $\pi_{(2,1,1)}$ with the multiplication map that collapses together the last two rows of a block in $U^{(2,1)}_{(2,1,1)}$, then we obtain the map $2 \cdot \pi_{(2,2)}$.

Definition 3.2.7 (Generic multi-prolongations)**.** We write $I_\mu = I^{\underline{d}}_{\mu}$ for the kernel of π_μ, and $I_r = I^{\underline{d}}_{r}$ for the intersection of the kernels of the maps π_μ, as μ ranges over partitions of r with at most (exactly) k parts. We refer to I_r as the set of "generic equations" for $\sigma_k(SV_{\underline{d}}(\mathbb{P}V_1^* \otimes \cdots \otimes \mathbb{P}V_n^*))$, or "generic multi-prolongations" (see Proposition 3.1.4).

3.2.2 Tableaux

The maps π_μ, for various partitions μ, are $S_{\underline{r}}$-equivariant, so to understand them it suffices to analyze them irreducible representation by irreducible representation. Recall that irreducible $S_{\underline{r}}$-representations are classified by n-partitions $\lambda \vdash^n \underline{r}$, so we fix one such. This gives rise to a Young symmetrizer c_λ as explained in Section 2.3, and all the data of π_μ (concerning the λ-parts of its kernel and image) is contained in its restriction to the λ-highest weight spaces of the source and target, i.e. in the map

$$\pi_\mu = \pi_\mu(\lambda) : c_\lambda \cdot U_r^{\underline{d}} \longrightarrow c_\lambda \cdot U_\mu^{\underline{d}}.$$

We now introduce the tableaux formalism that's fundamental for the proof of our main results, giving a combinatorial perspective on the analysis of the kernels and images of the maps π_μ, which are the main objects we're after.

The representations $U_\mu^{\underline{d}}$ are spanned by blocks M as in Definition 3.2.4, hence the vector spaces $c_\lambda \cdot U_\mu^{\underline{d}}$ are spanned by elements of the form $c_\lambda \cdot M$, which we shall represent as n-tableaux, according to the following definition.

Definition 3.2.8. Given a partition $\mu = (\mu_1, \cdots, \mu_t) \vdash r$, an n-partition $\lambda \vdash^n \underline{r}$ and a block $M \in U_\mu^{\underline{d}}$, we associate to the element $c_\lambda \cdot M \in c_\lambda \cdot U_\mu^{\underline{d}}$ the n-tableaux

$$T = (T^1, \cdots, T^n) = T^1 \otimes \cdots \otimes T^n$$

of shape λ, obtained as follows. Suppose that the block M has the set α_j^i in its i-th row and j-th column. Then we set equal to i the entries in the boxes of T^j indexed by elements of α_j^i (recall from Section 2.3 that the boxes of a tableau are indexed canonically: from left to right and top to bottom). Note that each tableaux T^j has entries $1, \cdots, t$, with i appearing exactly $\mu_i \cdot d_j$ times.

Note also that in order to construct the n-tableau T we have made a choice of the ordering of the rows of M: interchanging rows i and i' when $\mu_i = \mu_{i'}$ should yield the same element $M \in U_\mu^{\underline{d}}$, therefore we identify the corresponding n-tableaux that differ by interchanging the entries equal to i and i'.

Example 3.2.9. We let $n = 2, \underline{d} = (2, 1), r = 4, \mu = (2, 2)$ as in Example 3.1.2, and consider the 2-partition $\lambda = (\lambda^1, \lambda^2)$, with $\lambda^1 = (5,3)$, $\lambda^2 = (2,1,1)$. We have

$$
\begin{array}{ccc}
c_\lambda \cdot \begin{ytableau} 1,6 & 1 \\ 2,3 & 4 \\ 4,5 & 2 \\ 7,8 & 3 \end{ytableau} & = & \begin{ytableau} 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 4 \\ 2 \end{ytableau} \\
\Vert & & \Vert \\
c_\lambda \cdot \begin{ytableau} 2,3 & 4 \\ 7,8 & 3 \\ 1,6 & 1 \\ 4,5 & 2 \end{ytableau} & = & \begin{ytableau} 3 & 1 & 1 & 4 & 4 \\ 3 & 2 & 2 \end{ytableau} \otimes \begin{ytableau} 3 & 4 \\ 2 \\ 1 \end{ytableau}
\end{array}
$$

Let's write down the action of the map π_μ on the tableaux pictured above

$$
\pi_\mu \left(\begin{ytableau} 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 4 \\ 2 \end{ytableau} \right) = \begin{ytableau} 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 2 \\ 1 \end{ytableau} + \begin{ytableau} 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 \\ 2 \end{ytableau}
$$
$$
+ \begin{ytableau} 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 1 \\ 2 \end{ytableau} .
$$

We collect in the following lemma the basic relations that n-tableaux satisfy.

Lemma 3.2.10. *Fix an n partition $\lambda \vdash^n \underline{r}$, and let T be an n-tableau of shape λ. The following relations hold:*

1. *If σ is a permutation of the entries of T that preserves the set of entries in each column of T, then*
$$\sigma(T) = \mathrm{sgn}(\sigma) \cdot T.$$
In particular, if T has repeated entries in a column, then $T = 0$.

2. *If σ is a permutation of the entries of T that interchanges columns of the same size of some tableau T^j, then*
$$\sigma(T) = T.$$

3. *Assume that one of the tableaux of T, say T^j has a column C of size t with entries $a_1, a_2, \cdots, a_t$, and that b is an entry of T^j to the right of C. Let σ_i denote the transposition that interchanges a_i with b. We have*
$$T = \sum_{i=1}^{t} \sigma_i(T).$$

We write this as

$$\begin{array}{|c|c|}\hline a_1 & b \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_i & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \sum_{i=1}^{t} \begin{array}{|c|c|}\hline a_1 & a_i \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

disregarding the entries of T that don't get perturbed.

Proof. (1) follows from the fact that if $\sigma \in C_\lambda$ is a column permutation, then $b_\lambda \cdot \sigma = -b_\lambda$.

(2) follows from the fact that if σ permutes columns of the same size, then $\sigma \in R_\lambda$ is a permutation that preserves the rows of the canonical n-tableau of shape λ (so in particular $a_\lambda \cdot \sigma = a_\lambda$), and σ commutes with b_λ. It follows that

$$c_\lambda \cdot \sigma = a_\lambda \cdot (b_\lambda \cdot \sigma) = a_\lambda \cdot (\sigma \cdot b_\lambda) = (a_\lambda \cdot \sigma) \cdot b_\lambda = a_\lambda \cdot b_\lambda = c_\lambda.$$

(3) follows from Corollary 3.2.16 (note the rest of the proof uses the formalism of Section 3.2.3). Let us assume first that all entries $a_1, \cdots, a_t, b$ are distinct. If $\tilde{T}$ is the n-tableau obtained by circling the entries $a_1, \cdots, a_t, b$, then

$$\tilde{T} = \begin{array}{|c|c|}\hline a_1 & b \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_i & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} - \sum_{i=1}^{t} \begin{array}{|c|c|}\hline a_1 & a_i \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

By skew-symmetry on columns (part (1)), the effect of circling t entries in the same column of a tableau T is precisely multiplying T by $t!$. It follows that we can rewrite the above relation as

$$\tilde{T} = t! \cdot \left(\begin{array}{|c|c|}\hline a_1 & b \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_i & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} - \sum_{i=1}^{t} \begin{array}{|c|c|}\hline a_1 & a_i \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right).$$

By Corollary 3.2.16, $\tilde{T} = 0$, which combined with the above equality yields the desired relation.

Now if $a_1, \cdots, a_t, b$ are not distinct, then either $a_i = a_j$ for some $i \neq j$, or $b = a_i$ for some i. If $a_i = a_j$, then T and $\sigma_k(T)$, $k \neq i, j$, have repeated entries in the column C, hence they are zero. Relation (3) becomes then $0 = \sigma_i(T) + \sigma_j(T)$. But this is true by part (1), because $\sigma_i(T)$ and $\sigma_j(T)$ differ by a column transposition.

Assume now that $b = a_i$ for some i. Then $\sigma_j(T)$ has repeated entries in the column C for $j \neq i$, thus relation (3) becomes $T = \sigma_i(T)$, which is true because $a_i = b$. $\square$

There is one last ingredient that we need to introduce in the generic setting, namely the generic flattenings.

3.2.3 Generic flattenings

Definition 3.2.11 (Generic flattenings)**.** For a decomposition $\underline{d} = A + B$, $A = (a_1, \cdots, a_n)$, $B = (b_1, \cdots, b_n)$, (i.e. $d_i = a_i + b_i$ for $i = 1, \cdots, n$), we write $F_{A,B}^{k,r}$ for the span of $k \times k$ minors of generic (A, B)-flattenings. This is the subspace of $U_r^{\underline{d}}$ spanned by expressions of the form

$$[\alpha^1, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r},$$

where $[\alpha^1, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] = \det(z_{\alpha^i \cup \beta^j})$, $\alpha^i = (\alpha_1^i, \cdots, \alpha_n^i)$, $\beta^i = (\beta_1^i, \cdots, \beta_n^i)$, $\gamma^i = (\gamma_1^i, \cdots, \gamma_n^i)$, with $|\alpha_j^i| = a_j$, $|\beta_j^i| = b_j$ and $|\gamma_j^i| = d_j$, and such that for fixed j, the sets $\alpha_j^i, \beta_j^i, \gamma_j^i$ form a partition of the set $\{1, \cdots, rd_j\}$.

Example 3.2.12. Take $n = 2$, $\underline{d} = (2, 1)$ and $r = 4$, as usual. Take $A = (1, 1)$, $B = (1, 0)$ and $k = 3$. A typical element of $F_{A,B}^{3,4}$ looks like

$$D = [(\{1\}, \{1\}), (\{3\}, \{4\}), (\{7\}, \{3\}) | (\{6\}, \{\}), (\{2\}, \{\}), (\{8\}, \{\})] \cdot z_{(\{4,5\},\{2\})} =$$

$$\det \begin{bmatrix} z_{(\{1,6\},\{1\})} & z_{(\{1,2\},\{1\})} & z_{(\{1,8\},\{1\})} \\ z_{(\{3,6\},\{4\})} & z_{(\{3,2\},\{4\})} & z_{(\{3,8\},\{4\})} \\ z_{(\{7,6\},\{3\})} & z_{(\{7,2\},\{3\})} & z_{(\{7,8\},\{3\})} \end{bmatrix} \cdot z_{(\{4,5\},\{2\})}.$$

Expanding the determinant, we obtain

$$D = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 3,2 & 4 \\ \hline 7,8 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 3,6 & 4 \\ \hline 7,8 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,8 & 1 \\ \hline 3,2 & 4 \\ \hline 7,6 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 3,8 & 4 \\ \hline 7,2 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,8 & 1 \\ \hline 3,6 & 4 \\ \hline 7,2 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 3,8 & 4 \\ \hline 7,6 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array}.$$

Notice that all the blocks in the above expansion coincide, except in the entries $2, 6, 8$ that get permuted in all possible ways. Let's multiply now D with the Young symmetrizer c_λ for $\lambda = (\lambda^1, \lambda^2)$, $\lambda^1 = (5, 3)$ and $\lambda^2 = (2, 1, 1)$. We get

$$c_\lambda \cdot D = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 4 & 4 \\ \hline 1 & 3 & 3 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 4 & 4 \\ \hline 3 & 3 & 1 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

$$-\begin{ytableau} 1 & 3 & 2 & 4 & 4 \\ 1 & 3 & 2 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau} + \begin{ytableau} 1 & 3 & 2 & 4 & 4 \\ 2 & 3 & 1 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 4 & 4 \\ 3 & 3 & 2 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau}.$$

Note that all the 2-tableaux in the previous expression coincide, except in the 2-nd, 6-th and 8-th box of their first tableau, which get permuted in all possible ways. We represent $c_\lambda \cdot D$ by a 2-tableau with the entries in boxes 2, 6 and 8 of its first tableau circled (see also Definition 3.2.13 below):

$$c_\lambda \cdot D = \begin{ytableau} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau}.$$

To reformulate this one last time, we write

$$\begin{ytableau} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau} = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \cdot \sigma \left(\begin{ytableau} 1 & 2 & 2 & 4 & 4 \\ 1 & 3 & 3 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau} \right),$$

where $S_3 = S_{\{1,2,3\}}$ is the symmetric group on the circled entries.

Definition 3.2.13. Let A, B and $F^{k,r}_{A,B}$ as in Definition 3.2.11, let

$$D = [\alpha^1, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F^{k,r}_{A,B},$$

and let $\lambda \vdash^n \underline{r} = (rd_1, \cdots, rd_n)$. We let $\gamma^i = \alpha^i \cup \beta^i$ for $i = 1, \cdots, k$, and consider $T = c_\lambda \cdot m$ the n-tableau corresponding to the monomial

$$m = z_{\gamma^1} \cdots z_{\gamma^r}.$$

We represent $c_\lambda \cdot D \in \mathrm{hwt}_\lambda(F^{k,r}_{A,B})$ as the n-tableau T with the entries in the boxes corresponding to the elements of $\alpha^1, \cdots, \alpha^k$ circled. Alternatively, we can circle the entries in the boxes corresponding to the elements of $\beta^1, \cdots, \beta^k$.

It follows that a spanning set for $\mathrm{hwt}_\lambda(F^{k,r}_{A,B})$ can be obtained as follows: take all the subsets $\mathcal{C} \subset \{1, \cdots, r\}$ of size k, and consider all the n-tableaux T with a_j (alternatively b_j) of each of the elements of $\mathcal{C}$ circled in T^j. Of course, because of the symmetry of the alphabet $\{1, \cdots, r\}$ (see Definition 5.1.1), it's enough to only consider $\mathcal{C} = \{1, \cdots, k\}$, so that the only entries we ever circle are $1, 2, \cdots, k$.

Continuing with Example 3.2.12, we have

$$c_\lambda \cdot D = \begin{ytableau} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 3 \\ 2 \end{ytableau} = \begin{ytableau} \textcircled{1} & 2 & \textcircled{2} & 4 & 4 \\ 1 & \textcircled{3} & 3 \end{ytableau} \otimes \begin{ytableau} \textcircled{1} & 4 \\ \textcircled{3} \\ \textcircled{2} \end{ytableau}.$$

Our goal is to reduce the statement of Theorem 4.1.1 to an equivalent statement that holds in the generic setting, and thus transform our problem into a combinatorial one.

More precisely, we would like to say that the space of generic flattenings coincides with the intersection of the kernels of the (generic) maps π_μ, and that this is enough to conclude the same about the nongeneric case. One issue that arises is that we don't know at this point (although it seems very tempting to assert) that the zero-weight space of the space of flattenings coincides with the space of generic flattenings. Section 3.3 will show how to take care of this issue, and how to reduce all our questions to the generic setting.

3.2.4 1-flattenings

In this section we focus on the space of generic 1-flattenings, $F_1 = F_1^{k,r}$, defined as the subspace of $U_r^{\underline{d}}$ given by

$$F_1^{k,r} = \sum_{\substack{A+B=\underline{d} \\ |A|=1}} F_{A,B}^{k,r}.$$

We shall see that F_1 has a very simple representation theoretic description, which by the results of the next section will carry over to the nongeneric case.

Proposition 3.2.14. *With the above notations, we have*

$$F_1 = \bigoplus_{\substack{\lambda \vdash^n \underline{r} \\ \lambda_k \neq 0}} (U_r^{\underline{d}})_\lambda,$$

where $(U_r^{\underline{d}})_\lambda$ *denotes the* λ*-part of the representation* $U_r^{\underline{d}}$*, and* $\lambda_k \neq 0$ *means* $\lambda_k^j \neq 0$ *for some* $j = 1, \cdots, n$*, i.e. some partition* λ^j *has at least* k *parts.*

Proof. We divide the proof into two parts:

a) If $\lambda \vdash^n \underline{r}$ is an n-partition with some λ^j having at least k parts, and T is an n-tableau of shape λ, then $T \in F_1$.

b) If $\lambda \vdash^n \underline{r}$ is an n-partition with all λ^j having less than k parts, then $c_\lambda \cdot F_1 = 0$.

Let us start by proving part a). We assume that λ^j has at least k parts and consider T an n-tableau of shape λ. If T^j has repeated entries in its first column, then $T = 0$. Otherwise, we may assume that the first column of T^j has entries $1, 2, \cdots, t$ in this order, where t is the number of parts of λ^j, $t \geq k$. We consider the n-tableau $\tilde{T}$ obtained from T by circling the entries $1, 2, \cdots, k$ in the first column of T^j. We have

$$\tilde{T} = T^1 \otimes \cdots \otimes \begin{array}{|c|c|} \hline 1 & \cdots \\ \hline 2 & \cdots \\ \hline \vdots & \vdots \\ \hline k & \cdots \\ \hline k+1 & \cdots \\ \hline \vdots & \\ \hline \end{array} \otimes \cdots \otimes T^n,$$

i.e.

$$\tilde{T} = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \sigma(T),$$

where S_k denotes the symmetric group on the circled entries. Since $\sigma(T)$ differs from T by the column permutation σ, it follows by the skew-symmetry of tableaux that

$$\sigma(T) = \operatorname{sgn}(\sigma) \cdot T.$$

This shows that

$$\tilde{T} = k! \cdot T \Longleftrightarrow T = \frac{1}{k!} \cdot \tilde{T} \in F_1,$$

proving a).

To prove b), let

$$D = [\alpha^1, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F^{k,r}_{A,B},$$

for some $A + B = \underline{d}$ with $|A| = 1$. We have that $b_\lambda \cdot D$ is a linear combination of terms that look like D, so in order to prove that $c_\lambda = a_\lambda \cdot b_\lambda$ annihilates D, it suffices to show that $a_\lambda \cdot D = 0$.

We have $A = (a_1, \cdots, a_n)$ with $a_j = 1$ for some j and $a_i = 0$ for $i \neq j$. We can thus think of each of $\alpha^1, \cdots, \alpha^k$ as specifying a box in the partition λ^j. Since λ^j has less than k parts, it means that two of these boxes, say p and q, lie in the same same row of λ^j. Let $\sigma = (p, q)$ be the transposition of the two boxes. σ is an element in the group R_λ of permutations that preserve the rows of the canonical n-tableau of shape λ (Section 2.3), which means that $a_\lambda \cdot \sigma = a_\lambda$. However,

$$\begin{aligned} \sigma \cdot [\alpha^1, \cdots, \alpha^p, \cdots, \alpha^q, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] &= [\alpha^1, \cdots, \alpha^q, \cdots, \alpha^p, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] \\ &= -[\alpha^1, \cdots, \alpha^p, \cdots, \alpha^q, \cdots, \alpha^k | \beta^1, \cdots, \beta^k], \end{aligned}$$

since interchanging two rows/columns of a matrix changes the sign of its determinant. We get

$$a_\lambda \cdot D = (a_\lambda \cdot \sigma) \cdot D = a_\lambda \cdot (\sigma \cdot D) = a_\lambda \cdot (-D) = -a_\lambda \cdot D,$$

hence $a_\lambda \cdot D = 0$, as desired. $\square$

Remark 3.2.15. The nongeneric 1-flattenings give the equations of the so-called subspace varieties (see [Lan] or [Wey03, Prop. 7.1.2]), and in fact this statement is essentially equivalent to our Proposition 3.2.14 via the results of the next section, namely Proposition 3.3.5.

Corollary 3.2.16. *Let $\mathcal{C} \subset \{1, \cdots, r\}$ be a subset of size k. If λ is an n-partition with each λ^j having less than k parts, and $\tilde{T}$ is an n-tableau of shape λ, with one of each entries of $\mathcal{C}$ in $\tilde{T}^j$ circled, then $\tilde{T} = 0$. More generally, with no assumptions on λ, if the circled entries in $\tilde{T}^j$ all lie in columns of size less than k, then $\tilde{T} = 0$.*

Proof. The first part follows directly from Proposition 3.2.14, since $\tilde{T}$ is a 1-flattening, and the space of 1-flattenings doesn't have nonzero λ-parts when λ is such that each of its partitions have less than k parts.

For the more general statement, we can apply the argument for part b) of the proof of the previous proposition. If

$$D = [\alpha^1, \cdots, \alpha^k | \beta^1, \cdots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F_1$$

is such that each α^i corresponds to a box of $\tilde{T}^j$ situated in a column of size less than k, then since column permutations don't change the columns of the boxes corresponding to the α^i's, it follows that $b_\lambda \cdot D$ is a combination of expressions D' with the same properties as D. To show that $c_\lambda \cdot D = 0$ is thus enough to prove that $a_\lambda \cdot D = 0$. The proof of this statement is identical to the one in the preceding proposition. $\square$

The reader might wish to skip to Chapter 5 now, to get some familiarity with the tableaux combinatorics in the way it's going to be used throughout this work.

3.3 Polarization and specialization

In this section $V_1, \cdots, V_n$ are again vector spaces of arbitrary dimensions, $\dim(V_j) = m_j$, $j = 1, \cdots, n$. Let $\underline{r} = (r_1, \cdots, r_n)$ be a sequence of positive integers, and let

$$W = V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n}.$$

Let $S_{\underline{r}}$ denote the product of symmetric groups $S_{r_1} \times \cdots \times S_{r_n}$, and let $G \subset S_{\underline{r}}$ be a subgroup. Consider the natural (right) action of $S_{\underline{r}}$ on W obtained by letting S_{r_i} act by permuting the factors of $V_i^{\otimes r_i}$. More precisely, we write the pure tensors in W as

$$v = \bigotimes_{i,j} v_{ij}, \quad \text{with } v_{ij} \in V_j,\ j = 1, \cdots, n,\ i = 1, \cdots, r_j,$$

and for an element $\sigma = (\sigma^1, \cdots, \sigma^n) \in S_{\underline{r}}$, we let

$$v * \sigma = \bigotimes_{i,j} v_{\sigma^j(i)j}.$$

This action commutes with the (left) action of $GL(V)$ on W, and restricts to an action of G on W. It follows that W^G is a $GL(V)$-subrepresentation of W.

Proposition 3.3.1. *Continuing with the above notation, we let $U = W^G$ and $U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1})$. Let $\lambda \vdash^n \underline{r}$ be an n-partition with λ^j having at most m_j parts. The multiplicity of $S_\lambda V$ in U is the same with that of $[\lambda]$ in U'.*

Moreover, there exist polarization *and* specialization *maps*

$$P_\lambda : \mathrm{wt}_\lambda(U) \longrightarrow U', \quad Q_\lambda : U' \longrightarrow \mathrm{wt}_\lambda(U),$$

with the following properties:

1. *Q_λ is surjective.*
2. *P_λ is a section of Q_λ.*
3. *P_λ and Q_λ restrict to maps between $\mathrm{hwt}_\lambda(U)$ and $\mathrm{hwt}_\lambda(U')$ which are inverse to each other.*

Proof. The first part is a consequence of Schur-Weyl duality (Lemma 2.3.1) and Frobenius reciprocity (Lemma 2.3.3). We start with the identification

$$U = W^G = \mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}(W)).$$

Using Schur-Weyl duality we get

$$W = V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n} = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda] \otimes S_\lambda V,$$

therefore the previous equality becomes

$$U = \bigoplus_{\lambda \vdash^n \underline{r}} \mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}([\lambda])) \otimes S_\lambda V.$$

Frobenius reciprocity now yields

$$\mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}([\lambda])) = \mathrm{Hom}_{S_{\underline{r}}}(\mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1}), [\lambda]) = \mathrm{Hom}_{S_{\underline{r}}}(U', [\lambda]).$$

We get

$$U = \bigoplus_{\lambda \vdash^n \underline{r}} \mathrm{Hom}_{S_{\underline{r}}}(U', [\lambda]) \otimes S_\lambda V,$$

hence the multiplicity of $S_\lambda V$ in U coincides with that of $[\lambda]$ in U', as long as $S_\lambda V \neq 0$, i.e. as long as m_j is at least as large as the number of parts of the partition λ^j.

It follows that the vector spaces $\mathrm{hwt}_\lambda(U)$ and $\mathrm{hwt}_\lambda(U')$ have the same dimension, equal to the multiplicity of $S_\lambda V$ and $[\lambda]$ in U and U' respectively. We next construct explicit maps P_λ, Q_λ inducing isomorphisms of vector spaces between the two spaces.

We identify an element $\sigma = (\sigma^1, \cdots, \sigma^n) \in S_{\underline{r}}$ with the "tensor"

$$\bigotimes_{i,j} \sigma^j(i),$$

and consider the (regular) representation of $S_{\underline{r}}$ on the vector space R with basis consisting of the tensors σ for $\sigma \in S_{\underline{r}}$. The left action of $S_{\underline{r}}$ on R is given by

$$\sigma \cdot \bigotimes_{i,j} a_{ij} = \bigotimes_{i,j} \sigma^j(a_{ij}),$$

while the right action is given by

$$\bigotimes_{i,j} a_{ij} * \sigma = \bigotimes_{i,j} a_{\sigma^j(i)j}.$$

We consider the vector space map $Q_\lambda : R \to W$ given by

$$\bigotimes_{i,j} a_{ij} \longrightarrow \bigotimes_{i,j} g_j(a_{ij}),$$

where $g_j : \{1, \cdots, r_j\} \to \mathcal{B}_j$ is the map sending a to x_{ij} if the a-th box of λ^j is contained in the i-th row of λ^j (or equivalently if $\lambda_1^j + \cdots + \lambda_{i-1}^j < a \leq \lambda_1^j + \cdots + \lambda_i^j$). The image of Q_λ is $\mathrm{wt}_\lambda(W)$. It is clear that if $a = \bigotimes_{i,j} a_{ij}$ and $b = \bigotimes_{i,j} b_{ij}$, then $Q_\lambda(a) = Q_\lambda(b)$ if and only if $a = \sigma \cdot b$ for $\sigma \in S_{\underline{r}}$ a permutation that preserves the rows of the canonical n-tableau of shape λ. It follows that we can define $P_\lambda : \mathrm{wt}_\lambda(W) \to R$ by

$$P_\lambda(Q_\lambda(a)) = \frac{1}{\lambda!} a_\lambda \cdot a,$$

where a_λ is the row symmetrizer defined in Section 2.3, hence P_λ is a section of Q_λ.

Notice that P_λ and Q_λ are maps of right $S_{\underline{r}}$-modules, i.e. they respect the $*$-action of $S_{\underline{r}}$ on R and $\mathrm{wt}_\lambda(W)$ respectively.

Let us prove now that P_λ and Q_λ restrict to inverse isomorphisms between $\mathrm{hwt}_\lambda(R) = c_\lambda \cdot R$ (recall from Section 2.3 that c_λ denotes the Young symmetrizer corresponding to λ) and $\mathrm{hwt}_\lambda(W)$. The two spaces certainly have the same dimension (take $G = \{e\}$ to be the trivial subgroup of $S_{\underline{r}}$ and apply the first part of the proposition), so it's enough to prove that for $a' \in \mathrm{hwt}_\lambda(R)$

a) $Q_\lambda(a') \in \mathrm{hwt}_\lambda(W)$, and

b) $P_\lambda(Q_\lambda(a')) = a'$.

To see why part b) is true, note that

$$P_\lambda(Q_\lambda(a_\lambda \cdot a)) = \frac{1}{\lambda!} \cdot a_\lambda^2 \cdot a = a_\lambda \cdot a,$$

i.e. $P_\lambda \circ Q_\lambda$ fixes $a_\lambda \cdot R$. Since $\mathrm{hwt}_\lambda(R) = c_\lambda \cdot R \subset a_\lambda \cdot R$, it follows that $P_\lambda(Q_\lambda(a')) = a'$. To prove a) we need to show that $Q_\lambda(a')$ is fixed by the Borel (recall the definition of the Borel subgroup from 2.3). It's enough to do this when

$$a' = c_\lambda \cdot a, \quad a = \bigotimes_{i,j} a_{ij}.$$

The pure tensor a corresponds to an element $\sigma \in S_{\underline{r}}$, so we can write $a = e * \sigma$, where

$$e = \bigotimes_{i,j} e_{ij}$$

is the "identity" tensor, $e_{ij} = i$ for all i, j. It follows that

$$Q_\lambda(a') = Q_\lambda(c_\lambda \cdot a) = Q_\lambda(a_\lambda \cdot b_\lambda \cdot e * \sigma) = \lambda! \cdot Q_\lambda(b_\lambda \cdot e) * \sigma.$$

Since the $*$ action commutes with the action of the Borel, it is then enough to prove that $Q_\lambda(b_\lambda \cdot e)$ is fixed by the Borel. But this is a direct computation:

$$Q_\lambda(b_\lambda \cdot e) = \bigotimes_{i,j} x_{1j} \wedge \cdots \wedge x_{(\lambda^j)'_i j},$$

where $(\lambda^j)'$ denotes the conjugate partition of λ^j, so that in fact $(\lambda^j)'_i$ denotes the number of entries in the i-th column of λ^j. In any case, it is clear from the formula of $Q_\lambda(b_\lambda \cdot e)$ that it is invariant under the Borel, proving the claim that P_λ and Q_λ restrict to inverse isomorphisms between $\mathrm{hwt}_\lambda(W)$ and $\mathrm{hwt}_\lambda(R)$.

To finish the proof of the proposition, it suffices to notice that, by Remark 2.3.2, we have the identities

$$U = W^G = W * s \quad \text{and} \quad U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1}) = R * s,$$

where

$$s = \sum_{g \in G} g.$$

Now since P_λ, Q_λ respect the $*$ action, it follows that they restrict to inverse isomorphisms between

$$\mathrm{hwt}_\lambda(W) * s = \mathrm{hwt}_\lambda(W * s) = \mathrm{hwt}_\lambda(U)$$

and

$$\mathrm{hwt}_\lambda(R) * s = \mathrm{hwt}_\lambda(R * s) = \mathrm{hwt}_\lambda(U'),$$

proving the last part of the proposition. □

We shall apply Proposition 3.3.1 with $\underline{r} = (rd_1, \cdots, rd_n)$ and

$$U = U_r^{\underline{d}}(V) = S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n),$$

or more generally

$$U = U_\mu^{\underline{d}}(V) = \bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n),$$

the source and target respectively of the map π_μ in Definition 3.1.1. W is now the representation

$$W = V_1^{\otimes rd_1} \otimes \cdots \otimes V_n^{\otimes rd_n}.$$

We start with $U = U_r^{\underline{d}}(V)$. We have $U = W^G$ where

$$G = (S_{d_1} \times \cdots \times S_{d_n})^r \wr S_r$$

is the wreath product between $(S_{d_1} \times \cdots \times S_{d_n})^r$ and S_r. Recall that for a group H and positive integer r, the wreath product $H^r \wr S_r$ of H^r with the symmetric group S_r is just the semidirect product

$$H^r \rtimes S_r,$$

where S_r acts on H^r by permuting the r copies of H. We can thus identify an element $\sigma \in G$ with a collection

$$\sigma = ((\sigma_j^k)_{\substack{j=1,\cdots,n \\ k=1,\cdots,r}}, \tau),$$

where

$$\sigma_j^k \in S_{d_j}, \quad \tau \in S_r.$$

We need to say how we regard G as a subgroup of $S_{\underline{r}}$. First of all, we think of $S_{\underline{r}} = S_{rd_1} \times \cdots \times S_{rd_n}$ as a product of symmetric groups, where S_{rd_j} acts on the set $\mathcal{D}_j = \{1, \cdots, rd_j\}$. Then we think of an element $\sigma \in G$ as an element of $S_{\underline{r}}$ by letting σ_j^k act as a permutation of

$$\{(\tau(k) - 1) \cdot d_j + 1, \cdots, \tau(k) \cdot d_j\} \subset \mathcal{D}_j.$$

For example, when $d_1 = \cdots = d_n = 1$, G is just the group S_r, diagonally embedded in S_r^n. With this G, we let $U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1})$.

One can now see why the representation $U_r^{\underline{d}}$, as defined in the previous section, can be identified with U'. Recall that $U_r^{\underline{d}}$ was defined as a space of $r \times n$ blocks with certain identifications. Consider the block

$$M = \begin{array}{|c|c|c|c|} \hline \{1, \cdots, d_1\} & \{1, \cdots, d_2\} & \cdots & \{1, \cdots, d_n\} \\ \hline \{d_1 + 1, \cdots, 2d_1\} & \{d_2 + 1, \cdots, 2d_2\} & \cdots & \{d_n + 1, \cdots, 2d_n\} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \{(r-1)d_1 + 1, \cdots, rd_1\} & \{(r-1)d_2 + 1, \cdots, rd_2\} & \cdots & \{(r-1)d_n + 1, \cdots, rd_n\} \\ \hline \end{array}.$$

G acts trivially on M (because each σ_j^k does, and because the effect of τ is just permuting the rows of M), and all the other blocks are obtained from M by the action of some element of $S_{\underline{r}}$. One should think of the span of M thus as the trivial representation $\mathbf{1}$ of G that's induced to $S_{\underline{r}}$.

It is probably best to forget at this point that U' was the zero-weight space of a certain representation, and just think of it abstractly as the induced representation

$$\mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1}),$$

with its realization as a space of blocks. An important point to notice now is that for any decomposition $\underline{d} = A + B$, and any k, r we have

$$P_\lambda(\mathrm{wt}_\lambda(F^{k,r}_{A,B}(V))) \subset F^{k,r}_{A,B},$$

and

$$Q_\lambda(F^{k,r}_{A,B}) \subset \mathrm{wt}_\lambda(F^{k,r}_{A,B}(V)),$$

where $F^{k,r}_{A,B}$ (Definition 3.2.11) is the generic version of $F^{k,r}_{A,B}(V)$ (Definition 2.5.2). This means that on the corresponding λ-highest weight spaces, P_λ and Q_λ restrict to isomorphisms

$$\mathrm{hwt}_\lambda(F^{k,r}_{A,B}(V)) \simeq \mathrm{hwt}_\lambda(F^{k,r}_{A,B}).$$

Example 3.3.2. Here's an example of specialization, that involves blocks we're already familiar with. Let $n = 2$, $d_1 = 2$, $d_2 = 1$, $r = 4$, $\lambda^1 = (5, 3)$, $\lambda^2 = (2, 1, 1)$. The specialization map Q_λ sends

$$M = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 2,3 & 4 \\ \hline 4,5 & 2 \\ \hline 7,8 & 3 \\ \hline \end{array} \xrightarrow{Q_\lambda} \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 1,1 & 3 \\ \hline 1,1 & 1 \\ \hline 2,2 & 2 \\ \hline \end{array} = M'.$$

Q_λ sends $1, 2, 3, 4, 5$ from the first column of M to 1, because boxes $1, 2, 3, 4, 5$ of λ^1 lie in the first row of λ^1, and it sends $6, 7, 8$ to 2 because boxes $6, 7, 8$ of λ^1 lie in its second row. A similar description holds for the second column of M and λ^2.

Although we won't write down explicitly $P_\lambda(M')$ in this example (see the example below for a concrete illustration of the action of P_λ), we will just mention that $P_\lambda(M')$ is the average of the blocks that specialize to M' via the specialization map Q_λ. Of course, M is one such block, but there are many more others.

Example 3.3.3. Let $n = 3$, $d_1 = d_2 = d_3 = 1$ and $\lambda^1 = \lambda^2 = \lambda^3 = (2, 1)$. If $m = z_{(\{1\},\{1\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} \in U'$, then

$$Q_\lambda(m) = z_{(\{1\},\{1\},\{1\})} z_{(\{1\},\{2\},\{1\})} z_{(\{2\},\{1\},\{2\})} \in U$$

and

$$\begin{aligned} P_\lambda(Q_\lambda(m)) = \frac{1}{8}\big(& z_{(\{1\},\{1\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} + z_{(\{2\},\{1\},\{2\})} z_{(\{1\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} \\ & + z_{(\{1\},\{1\},\{1\})} z_{(\{2\},\{3\},\{2\})} z_{(\{3\},\{2\},\{3\})} + z_{(\{2\},\{1\},\{1\})} z_{(\{1\},\{3\},\{2\})} z_{(\{3\},\{2\},\{3\})} \\ & + z_{(\{1\},\{2\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{1\},\{3\})} + z_{(\{2\},\{2\},\{2\})} z_{(\{1\},\{3\},\{1\})} z_{(\{3\},\{1\},\{3\})} \\ & + z_{(\{1\},\{2\},\{1\})} z_{(\{2\},\{3\},\{2\})} z_{(\{3\},\{1\},\{3\})} + z_{(\{2\},\{2\},\{1\})} z_{(\{1\},\{3\},\{2\})} z_{(\{3\},\{1\},\{3\})}\big). \end{aligned}$$

When $U = U_\mu^{\underline{d}}(V)$, with $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$, we get $U = W^G$, where

$$G = \bigtimes_{j=1}^{s} \left((S_{\mu_j d_1} \times \cdots \times S_{\mu_j d_n})^{i_j} \wr S_{i_j} \right).$$

It follows that $U' = \mathrm{Ind}_G^{S_r}(\mathbf{1}) = U_\mu^{\underline{d}}$ with the realization as a space of blocks explained in the preceding section.

We note that the maps π_μ and $\pi_\mu(V)$ commute with the polarization and specialization maps P_λ, Q_λ, i.e. we have a commutative diagram

$$\begin{tikzcd}
U_r^{\underline{d}} \arrow[r, shift left, "Q_\lambda"] \arrow[d, "\pi_\mu"'] & \mathrm{wt}_\lambda(U_r^{\underline{d}}(V)) \arrow[l, shift left, "P_\lambda"] \arrow[d, "\pi_\mu(V)"] \\
U_\mu^{\underline{d}} \arrow[r, shift left, "Q_\lambda"] & \mathrm{wt}_\lambda(U_\mu^{\underline{d}}(V)) \arrow[l, shift left, "P_\lambda"]
\end{tikzcd} \tag{3.3.1}$$

Example 3.3.4. Let $\underline{d} = (2,1)$, $r = 4$, $\mu = (2,2)$, $\lambda^1 = (5,3)$, $\lambda^2 = (2,1,1)$. We only illustrate the specialization map Q_λ, with the above diagram transposed:

$$\begin{tikzcd}
\begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 2,3 & 4 \\ \hline 4,5 & 2 \\ \hline 7,8 & 3 \\ \hline \end{array} \arrow[r, "\pi_\mu"] \arrow[d, "Q_\lambda"'] & \begin{array}{|c|c|}\hline 1,2,3,6 & 1,4 \\ \hline 4,5,7,8 & 2,3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,4,5,6 & 1,2 \\ \hline 2,3,7,8 & 3,4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,6,7,8 & 1,3 \\ \hline 2,3,4,5 & 2,4 \\ \hline \end{array} \arrow[d, "Q_\lambda"] \\
\begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 1,1 & 3 \\ \hline 1,1 & 1 \\ \hline 2,2 & 2 \\ \hline \end{array} \arrow[r, "\pi_\mu(V)"] & \begin{array}{|c|c|}\hline 1,1,1,2 & 1,3 \\ \hline 1,1,2,2 & 1,2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,1,1,2 & 1,1 \\ \hline 1,1,2,2 & 2,3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,2,2,2 & 1,2 \\ \hline 1,1,1,1 & 1,3 \\ \hline \end{array}
\end{tikzcd}$$

Restricting 3.3.1 to the λ-highest weight spaces, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{hwt}_\lambda(U_r^{\underline{d}}) \arrow[r, shift left, "Q_\lambda"] \arrow[d, "\pi_\mu"'] & \mathrm{hwt}_\lambda(U_r^{\underline{d}}(V)) \arrow[l, shift left, "P_\lambda"] \arrow[d, "\pi_\mu(V)"] \\
\mathrm{hwt}_\lambda(U_\mu^{\underline{d}}) \arrow[r, shift left, "Q_\lambda"] & \mathrm{hwt}_\lambda(U_\mu^{\underline{d}}(V)) \arrow[l, shift left, "P_\lambda"]
\end{tikzcd}$$

where all the horizontal maps are isomorphisms. This shows that the λ-highest weight spaces of the kernels of π_μ and $\pi_\mu(V)$ get identified via the polarization and specialization maps, and therefore the same is true for I_r^d and $I_r^d(V)$: the generic multi-prolongations and multi-prolongations correspond to each other via polarization and specialization. We summarize the conclusions of this section in the following

Proposition 3.3.5. *The polarization and specialization maps P_λ and Q_λ restrict to maps between generic flattenings and flattenings, inducing inverse isomorphisms*

$$\mathrm{hwt}_\lambda(F_{A,B}^{k,r}) \simeq \mathrm{hwt}_\lambda(F_{A,B}^{k,r}(V)).$$

They also restrict to maps between the kernels of the generic π_μ's and the nongeneric ones, inducing inverse isomorphisms

$$\mathrm{hwt}_\lambda(\ker(\pi_\mu)) \simeq \mathrm{hwt}_\lambda(\ker(\pi_\mu(V))).$$

As a consequence, P_λ and Q_λ yield inverse isomorphisms between the λ-highest weight spaces of generic and nongeneric multi-prolongations

$$\mathrm{hwt}_\lambda(I_r^d) \simeq \mathrm{hwt}_\lambda(I_r^d(V)).$$

It follows that in order to show that flattenings coincide with multi-prolongations for the variety of secant lines to a Segre-Veronese variety (Theorem 4.1.1), it suffices to prove their equality in the generic setting.

Chapter 4

The secant line variety of a Segre-Veronese variety

This chapter is based on the techniques developed in the preceding one. We use the reduction to the "generic" situation to work out the analysis of the equations and coordinate rings of secant varieties of Segre-Veronese varieties in the first new interesting case, that of the first secant varieties. We show how in the case of the secant line variety $\sigma_2(X)$ of a Segre-Veronese variety X, the combinatorics of tableaux can be used to show that the "generic equations" coincide with the 3×3 minors of "generic flattenings". In particular, we confirm a conjecture of Garcia, Stillman and Sturmfels, which constitutes the special case when X is a Segre variety. We also obtain the representation theoretic description of the homogeneous coordinate ring of $\sigma_2(X)$, which in particular can be used to compute the Hilbert function of $\sigma_2(X)$. In the special cases when $\sigma_2(X)$ coincides with the ambient space, we obtain the decomposition into irreducible representations of certain plethystic compositions. Section 4.1 describes the statements of our results, while Section 4.2 contains the details of the proofs.

4.1 Main result and consequences

The main result of our thesis is the description of the generators of the ideal of the variety of secant lines to a Segre-Veronese variety, together with the decomposition of its coordinate ring as a sum of irreducible representations.

Theorem 4.1.1. *Let $X = SV_{d_1,\cdots,d_n}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ be a Segre-Veronese variety, where each V_i is a vector space of dimension at least 2 over a field K of characteristic zero. The ideal of $\sigma_2(X)$ is generated by 3×3 minors of flattenings, and moreover, for every nonnegative integer r we have the decomposition of the degree r part of its homogeneous* *coordinate ring*

$$K[\sigma_2(X)]_r = \bigoplus_{\substack{\lambda=(\lambda^1,\cdots,\lambda^n)\\ \lambda^i\vdash rd_i}} (S_{\lambda^1}V_1 \otimes \cdots \otimes S_{\lambda^n}V_n)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \max_{i=1,\cdots,n} \left\lceil \frac{\lambda_2^i}{d_i} \right\rceil, \quad e_\lambda = \lambda_2^1 + \cdots + \lambda_2^n.$$

If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

As a consequence, we derive the conjecture by Garcia, Stillman and Sturmfels, concerning the equations of the secant line variety of a Segre variety.

Corollary 4.1.2. *The GSS conjecture (Conjecture 1.2.1) holds, namely the ideal of the variety of secant lines to a Segre product of projective spaces is generated by 3×3 minors of flattenings.*

Proof. This is the special case of the first part of Theorem 4.1.1 when $d_1 = d_2 = \cdots = d_n = 1$. □

Combining Theorem 4.1.1 with known dimension calculations for secant varieties of Segre and Veronese varieties, we obtain two interesting plethystic formulas. We do not claim that these formulas are new: since all the vector spaces involved have dimension two, the representation theory of $\mathfrak{sl}_2$ can be also used to deduce them. However, we hope that the simple idea we present, together with a generalization of the last part of Theorem 4.1.1 to higher secant varieties, would yield new plethystic formulas for decomposing Schur functors applied to tensor products of representations.

Corollary 4.1.3. *a) Let V_1, V_2, V_3 be vector spaces of dimension two over a field K of characteristic zero, and let r be a positive integer. We have the decomposition*

$$\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3) = \bigoplus_{\substack{\lambda=(\lambda^1,\lambda^2,\lambda^3)\\ \lambda^i\vdash r}} (S_{\lambda^1}V_1 \otimes S_{\lambda^2}V_2 \otimes S_{\lambda^3}V_3)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \max\{\lambda_2^1, \lambda_2^2, \lambda_2^3\}, \quad e_\lambda = \lambda_2^1 + \lambda_2^2 + \lambda_2^3.$$

If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If

$e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

b) Let V_1, V_2 be vector spaces of dimension two over a field K of characteristic zero, let r be a positive integer and let $\mu = (\mu_1, \mu_2)$ be a partition of r with at most two parts. We have the decomposition

$$S_\mu(V_1 \otimes V_2) = \bigoplus_{\substack{\lambda=(\lambda^1,\lambda^2) \\ \lambda^i \vdash r}} (S_{\lambda^1}V_1 \otimes S_{\lambda^2}V_2)^{m_\lambda},$$

with $m_\lambda = m_{(\lambda^1,\lambda^2,\mu)}$, where $m_{(\lambda^1,\lambda^2,\mu)}$ is as defined in part a).

Proof. Part a) follows from the fact that the secant line variety of a 3-factor Segre variety X has the expected dimension, namely $2 \cdot \dim(X) + 1$. In the case we are interested in $X = \mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ has dimension 3 and is a subvariety of $\mathbb{P}^{2\cdot 2\cdot 2-1} = \mathbb{P}^7$, so $\sigma_2(X)$ fills in the whole space. This means that the coordinate ring of $\sigma_2(X)$ and $\mathbb{P}^7$ coincide, i.e.

$$K[\sigma_2(X)] = \mathrm{Sym}(V_1 \otimes V_2 \otimes V_3),$$

and therefore we can use the description of Theorem 4.1.1 to compute

$$K[\sigma_2(X)]_r = \mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3).$$

As for part b), let V_3 be another vector space of dimension two. Part a) tells us how to decompose

$$\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3)$$

in general. On the other hand, regarding $V_1 \otimes V_2 \otimes V_3$ as the tensor product between the vector spaces $V_1 \otimes V_2$ and V_3, we can use Cauchy's formula (Section 5.2) to obtain

$$\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3) = \bigoplus_{\mu \vdash r} S_\mu(V_1 \otimes V_2) \otimes S_\mu V_3.$$

Now the desired formula for the multiplicity of the irreducible representations occurring in $S_\mu(V_1 \otimes V_2)$ follows by combining the formula from part a) with the Cauchy formula depicted above. □

Corollary 4.1.4. *Let V be a vector space of dimension two over a field K of characteristic zero. We have the decomposition*

$$\mathrm{Sym}^r(\mathrm{Sym}^3(V)) = \bigoplus_{\lambda \vdash 3r} (S_\lambda V)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \left\lceil \frac{\lambda_2}{3} \right\rceil, \quad e_\lambda = \lambda_2.$$

If λ has more than two parts, or if $e_\lambda < 2f_\lambda$ (i.e. $\lambda_2 = 1$), then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

Proof. This follows from the fact that $\sigma_2(\mathrm{Ver}_3(\mathbb{P}^1))$, the secant line variety of the twisted cubic fills in the space, hence its coordinate ring is

$$\mathrm{Sym}(\mathrm{Sym}^3(V)).$$

Using the description in Theorem 4.1.1 with $n = 1$, $d_1 = 3$ and $V = V_1$ of dimension 2, we obtain the desired formula. □

4.2 Proof of the main result

Proof. We start by outlining the main steps in the proof of Theorem 4.1.1. We fix a sequence of positive integers $\underline{d} = (d_1, \cdots, d_n)$ and a positive degree r, and let $\underline{r} = (rd_1, \cdots, rd_n)$. By Proposition 3.3.5, it suffices to prove the generic version of the theorem. More precisely, we let

$$F = \sum_{A+B=\underline{d}} F^{3,r}_{A,B} \subset U^{\underline{d}}_r$$

be the set of generic flattenings, and let F_i denote those generic flattenings with $|A| = i$. (As the rest of the proof will imply, we have $F = F_1 + F_2 + F_3$; see Chapter 6 for more precise results in this direction in the case $n = 1$ of the Veronese variety.)

Recall that $I = I^{\underline{d}}_r$ denotes the space of generic multi-prolongations of degree r (Definition 3.2.7), i.e. I is the kernel of the map

$$\pi = \bigoplus_{\mu=(\mu_1,\mu_2)\vdash r} \pi_\mu : U = U^{\underline{d}}_r \longrightarrow \bigoplus_{\mu=(\mu_1,\mu_2)\vdash r} U^{\underline{d}}_\mu.$$

We have $F \subset I$, by combining Lemma 2.5.1 with Proposition 3.3.5. We will show that $F = I$ and that the image of π decomposes into irreducible $S_{\underline{r}}$-representations as

$$\pi(U) = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda]^{m_\lambda},$$

where m_λ is as defined in the statement of the theorem.

We list the main steps below. The details will occupy the rest of the chapter.

Step 0: If λ is an n-partition with some λ^i having at least three parts, then $\mathrm{hwt}_\lambda(U) = \mathrm{hwt}_\lambda(F)$ (Proposition 3.2.14), hence $\mathrm{hwt}_\lambda(F) = \mathrm{hwt}_\lambda(I)$, because $F \subset I \subset U$. Moreover, this also shows that $m_\lambda = 0$.

Step 1: We fix an n-partition λ of $\underline{r}$ with each λ^i having at most two parts. We identify each tableau T with a certain graph G. We show that graphs containing odd cycles are contained in F.

Step 2: We show that the λ-highest weight space of U/F is spanned by bipartite graphs that are as connected as possible, i.e. that are either connected, or a union of a tree and some isolated nodes.

Step 3: We introduce the notion of type associated to a graph G as in Step 2, encoding the sizes of the sets in the bipartition of the maximal component of G. We show that if G_1, G_2 have the same type, then $G_1 = \pm G_2$ (modulo F).

Step 4: If we let

$$\pi = \bigoplus_{\substack{\mu \vdash r \\ \mu=(a\geq b)}} \pi_\mu : U \longrightarrow \bigoplus_{\substack{\mu \vdash r \\ \mu=(a\geq b)}} U^n_\mu,$$

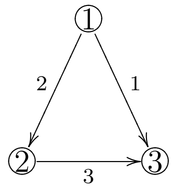
and if G_i are graphs of distinct types (not contained in F), then the elements $\pi(G_i)$ are linearly independent. This suffices to prove that F and the kernel of π are the same, i.e. that $F = I$. The formulas for the multiplicities m_λ follow from counting the number of G_i's, i.e. the number of possible types.

4.2.1 Step 1

We fix an n-partition λ of $\underline{r}$ with $\lambda^i = (\lambda^i_1 \geq \lambda^i_2 \geq 0)$, for $i = 1, \cdots, n$. For each n-tableau T of shape λ we construct a graph G with r vertices labeled by the elements of the alphabet $\mathcal{A} = \{1, \cdots, r\}$ as follows. For each tableau T^i of T and column $\begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array}$ of T^i of length 2, G has an oriented edge (x, y) which we label by the index i. We will often refer to the labels of the edges of G as colors. Note that we allow G to have multiple edges between two vertices (some call such G a multigraph), but at any given vertex there can be at most d_i incident edges of color i. Since we think of two n-tableaux as being the same if they differ by a permutation of $\mathcal{A}$, we shall also identify two graphs if they differ by a relabeling of their nodes. Note that a graph G determines an element in $\mathrm{hwt}_\lambda(U)$, by considering a tableau T with columns $\begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array}$ for each edge (x, y) of G. The order of the columns of T is not determined by G, but part (2) of Lemma 3.2.10 states that any such T yields the same element of $\mathrm{hwt}_\lambda(U)$. The orientation of the edges of our graphs will be mostly irrelevant: reversing the orientation of an edge of $G = T$ will correspond to changing G to $-G$ (see part (1) of Lemma 3.2.10). When we talk about connectedness and cycles, we don't take into account the orientation of the edges.

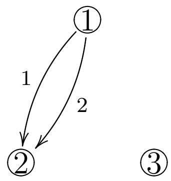
Example 4.2.1. The graph

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} =$$



is connected and has a cycle of length 3, while

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} =$$



is disconnected and has a cycle of length 2.

From now on we work modulo F, and more precisely, inside the λ-highest weight space of (U/F). This space is generated by the graphs described above. The main result of Step 1 is

Proposition 4.2.2. *If G has an odd cycle, then $G = 0$ (i.e. G is in F).*

We first need to establish some fundamental relations, that will be used throughout the rest of the proof.

Lemma 4.2.3. *The following relations between tableaux/graphs hold (see the interpretation below)*

a) $\begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array} = -\begin{array}{|c|}\hline y \\ \hline x \\ \hline \end{array}$, *in particular* $\begin{array}{|c|}\hline x \\ \hline x \\ \hline \end{array} = 0.$

b) $\begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline z & x \\ \hline y \\ \cline{1-1} \end{array}.$

c) $\begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array}.$

d) $\begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array}.$

Interpretation: For an expression $E = \sum_T a_T \cdot T$, where the T's are n-tableaux of shape λ, we say that $E = 0$ if

$$\sum_T a_T \cdot T \in \ F \subset U.$$

If all the n tableaux occurring in the expression E contain the same n-subtableau S, then we suppress S entirely from the notation (see also the comment in part (3) of Lemma 3.2.10).

Example 4.2.4. One interpretation of part b) of Lemma 4.2.3 could be that

$$\begin{ytableau} a & b \\ c & d \end{ytableau} \otimes \begin{ytableau} x & z & t \\ y \end{ytableau} = \begin{ytableau} a & b \\ c & d \end{ytableau} \otimes \begin{ytableau} x & y & t \\ z \end{ytableau} + \begin{ytableau} a & b \\ c & d \end{ytableau} \otimes \begin{ytableau} z & x & t \\ y \end{ytableau},$$

for any $\{a, b, c, d\} = \{x, y, z, t\} = \{1, 2, 3, 4\}$. The 2-subtableau S is in this case

$$S = \begin{ytableau} a & b \\ c & d \end{ytableau} \otimes \begin{ytableau} t \end{ytableau}.$$

Proof of Lemma 4.2.3. a) is part (1) of Lemma 3.2.10.

b) follows from part (3) of the same lemma (since all columns of our tableaux have size at most two).

c) We have

$$\begin{ytableau} x & y \\ z \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \cdot \sigma \left(\begin{ytableau} x & y \\ z \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} \right) = 0,$$

(because the left hand side is contained in F_2). Using parts a) and b) repeatedly, we can express everything in terms of $\begin{ytableau} x & y \\ z \end{ytableau}$ and $\begin{ytableau} x & z \\ y \end{ytableau}$, and after simplifications, the above equation becomes

$$3 \cdot \left(\begin{ytableau} x & y \\ z \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} - \begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau} \right) = 0.$$

d) Part c) states that any tensor expression in $a = \begin{ytableau} x & z \\ y \end{ytableau}$ and $b = \begin{ytableau} x & y \\ z \end{ytableau}$ does not depend on the order in which a and b appear, so we can think of the pure tensors in a, b as commuting monomials in a, b. Writing $\begin{ytableau} y & x \\ z \end{ytableau} = b - a$, we can translate

$$\begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau} = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \cdot \sigma \left(\begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau} \right) = 0,$$

into

$$a^2 b - a^2(b - a) + (a - b)^2 b - b^2 a - (b - a)^2 a + b^2(a - b) = 0,$$

which simplifies to $3(a^2 b - ab^2) = 0$, i.e. $a^2 b = ab^2$, or

$$\begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau} = \begin{ytableau} x & z \\ y \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau} \otimes \begin{ytableau} x & y \\ z \end{ytableau}.$$

□

Corollary 4.2.5. *If G is a graph having a connected component H consisting of two nodes joined by an odd number of edges, then $G = 0$.*

Proof. Interchanging the labels of the two nodes of H preserves G, but by part a) of Lemma 4.2.3, it also transforms G into $(-1)^e G$, where e is the number of edges in H. Since e is odd, $G = 0$. $\square$

Corollary 4.2.6. *If G is a graph containing cycles of length 1 or 3, then $G = 0$.*

Proof. If G has a cycle of length 1, this follows from part a) of Lemma 4.2.3. If G has a cycle of length 3, we may assume this cycle is $C = \textcircled{1} \to \textcircled{2} \to \textcircled{3} \to \textcircled{1}$. We have several cases to analyze, depending on the colors of the edges in this cycle.

If the edges in C have distinct colors, we need to prove that

$$\ytableaushort{13,2} \otimes \ytableaushort{21,3} \otimes \ytableaushort{12,3} = 0.$$

We have by part b) of Lemma 4.2.3 applied to the middle tableau that

$$\ytableaushort{13,2} \otimes \ytableaushort{21,3} \otimes \ytableaushort{12,3} = \ytableaushort{13,2} \otimes \ytableaushort{12,3} \otimes \ytableaushort{12,3} - \ytableaushort{13,2} \otimes \ytableaushort{13,2} \otimes \ytableaushort{12,3} = 0,$$

where the last equality is part d) of the same lemma.

If the edges of C have the same color, we need to prove that

$$\ytableaushort{112,233} = 0.$$

We have

$$0 = \ytableaushort{{\textcircled{1}}{\textcircled{1}}{\textcircled{2}},{\textcircled{2}}{\textcircled{3}}{\textcircled{3}}} = \ytableaushort{112,233} - \ytableaushort{221,133} - \ytableaushort{332,211} - \ytableaushort{113,322} + \ytableaushort{223,311} + \ytableaushort{331,122}$$

$$= \ytableaushort{112,233} + \ytableaushort{121,233} + \ytableaushort{211,332} + \ytableaushort{112,323} + \ytableaushort{211,323} + \ytableaushort{121,332} = 6 \cdot \ytableaushort{112,233},$$

where the penultimate equality follows from skew-symmetry on rows, while the last one follows from part (2) of Lemma 3.2.10.

Finally, suppose that the edges of C have two colors, say $(1,2)$ and $(1,3)$ have the same color. We need to prove that

$$\ytableaushort{1123,23} \otimes \ytableaushort{21,3} = 0.$$

As in the preceding case,

$$0 = \ytableaushort{{\textcircled{1}}{\textcircled{1}}{\textcircled{2}}{\textcircled{3}},{\textcircled{2}}{\textcircled{3}}} \otimes \ytableaushort{{\textcircled{2}}{\textcircled{1}},{\textcircled{3}}} = \ytableaushort{1123,23} \otimes \ytableaushort{21,3} - \ytableaushort{2213,13} \otimes \ytableaushort{12,3} - \ytableaushort{3321,21} \otimes \ytableaushort{23,1}$$

$$- \ytableaushort{1132,32} \otimes \ytableaushort{31,2} + \ytableaushort{2231,31} \otimes \ytableaushort{32,1} + \ytableaushort{3312,12} \otimes \ytableaushort{13,2} = 6 \cdot \ytableaushort{1123,23} \otimes \ytableaushort{21,3},$$

where the last equality follows by utilizing repeatedly parts a) and c) of 4.2.3. For example, we have for the second term that

$$\begin{array}{|c|c|c|c|}\hline 2&2&1&3\\ \hline 1&3 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&2&1&3\\ \hline 2&3 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 2&1\\ \hline 3 \\ \cline{1-1}\end{array},$$

where the last equality follows by applying part c) of 4.2.3 in the form

$$\begin{array}{|c|c|}\hline y&z\\ \hline x \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline z&y\\ \hline x \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline z&y\\ \hline x \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline y&z\\ \hline x \\ \cline{1-1}\end{array},$$

with

$$\begin{array}{|c|c|}\hline y&z\\ \hline x \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&1\\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline z&y\\ \hline x \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array}.$$

□

Corollary 4.2.7. *If an n-tableau T contains the columns $C_1 = \begin{array}{|c|}\hline x\\ \hline y\\ \hline\end{array}$ and $C_2 = \begin{array}{|c|}\hline x\\ \hline z\\ \hline\end{array}$, and T' is obtained from T by interchanging two boxes $\boxed{y}$ and $\boxed{z}$ from the same tableau T^i of T, and not contained in any of C_1, C_2, then $T = T'$ (modulo F).*

Proof. If $\begin{array}{|c|}\hline y\\ \hline z\\ \hline\end{array}$ is a column of T^i then T contains a triangle, hence $T = 0$. Since interchanging y and z transforms T into $T' = -T = 0$, it follows that $T = T'$. We can assume then that y and z don't lie in the same column of T^i. If they both belong to columns of size one of T^i, then interchanging them preserves T (see part (2) of Lemma 3.2.10). Otherwise we may assume that y belongs to a column of size two in T^i, hence we have the relation

$$\begin{array}{|c|c|}\hline y&z\\ \hline * \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline y&*\\ \hline z \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline z&y\\ \hline * \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline z&y\\ \hline * \\ \cline{1-1}\end{array},$$

where the last equality follows from the fact that any tableau containing C_1, C_2 and $\begin{array}{|c|}\hline y\\ \hline z\\ \hline\end{array}$ is a graph containing a triangle, i.e. it is zero (Corollary 4.2.6). □

Proof of Proposition 4.2.2. We show that a graph G (with corresponding tableau T) containing an odd cycle of length at least 5 is a linear combination of graphs with shorter odd cycles. The conclusion then follows by induction from Corollary 4.2.6. Suppose that $C : ① \to ② \to \cdots \to ⓚ \to ①$ is an odd cycle in G, with $k \geq 5$. We denote by E_i the edge $(i, i+1)$ ($E_k = (k, 1)$).

Let's assume first that there are two consecutive edges of C of the same color, say E_1 and E_2 have color 1. If not all edges of C have color 1, we may assume that E_3 has color 2, so that T contains the subtableau

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline\end{array} \otimes \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}.$$

Since E_1, E_2 have color 1, it follows that $d_1 \geq 2$, hence there are at least two 4's in T^1. One of them is thus not contained in E_5, and therefore in none of the edges of C. We apply Corollary 4.2.7 with C_1, C_2 the columns corresponding to E_2, E_3, $y = 2$ and $z = 4$. We can thus interchange the 2 in E_1 with a $4 \in T^1$ not in any E_i, obtaining an n-tableau $T' = T$, with T' containing the cycle $\textcircled{1} \to \textcircled{4} \to \textcircled{5} \to \cdots \to \textcircled{k} \to \textcircled{1}$ of length $k - 2$.

If all the E_i's have color 1, T contains the subtableau

$$S = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & \cdots & k \\ \hline 2 & 3 & 4 & 5 & \cdots & 1 \\ \hline \end{array} \subset T^1.$$

If there is an edge $(3,4)$ of G with color different from 1, then we can replace E_3 by that edge and apply the previous case. If $d_1 > 2$ then T^1 has a 4 not contained in any E_i, so we can again use the argument from the previous paragraph. Suppose now that $d_1 = 2$. The proof of Corollary 4.2.7 shows that we can interchange 3 with 4 in all T^i's $(i \neq 1)$, modulo tableaux containing S and an edge $(3,4)$ of color different from 1. But these we know are zero (modulo F) by the argument above, so we can write $T = T'$ where T' is obtained from T by interchanging all 3's and 4's in T^i for $i \geq 2$. We now use the relation

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 5 & 4 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 4 & 4 & 5 \\ \hline \end{array},$$

to write

$$T' = T'' + T''',$$

where T'' contains the cycle $\textcircled{1} \to \textcircled{2} \to \textcircled{5} \to \cdots \to \textcircled{k} \to \textcircled{1}$ of length $k-2$, and T''' differs from T by interchanging all the 3's and 4's in T, and doing a column transposition in the column of E_3. This shows that $T = T' = 0 - T$, hence $T = 0$.

Finally, we assume that no two consecutive edges have the same color. Since the cycle is odd, we can find three consecutive edges with distinct colors, say E_1, E_2 and E_3, with colors $1, 2$ and 3 respectively. By Corollary 4.2.7, we have

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}.$$

If the edge E_4 in C doesn't have color 1, then it survives after interchanging 2 and 4 as above, hence T is equal with a graph containing the odd cycle $\textcircled{1} \to \textcircled{4} \to \textcircled{5} \to \cdots \to \textcircled{k} \to \textcircled{1}$ of length $k - 2$.

Suppose now that E_4 has color 1. If the edge E_5 doesn't have color 2, then we may repeat the above argument replacing the edges E_1, E_2 and E_3 with E_2, E_3 and E_4 respectively. Otherwise, T contains the subtableau (with $* = 6$ if $k > 5$ and $* = 1$ if $k = 5$)

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & * \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & * \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & * \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 5 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array},$$

where the first equality follows by interchanging 2 and 4 in the first factor, while the last one follows by interchanging 3 and 5 in the last factor (in both cases we apply Corollary 4.2.7). It follows that T is equal to a graph containing the odd cycle ① → ④ → ⑤ → ⋯ → ⓚ → ① of length $k-2$, concluding the proof. $\square$

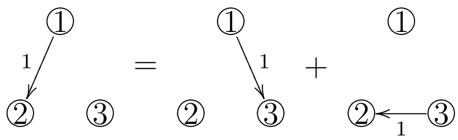
4.2.2 Step 2

We first translate the relations in part b) of Lemma 4.2.3 into basic operations on graphs. We start with the following

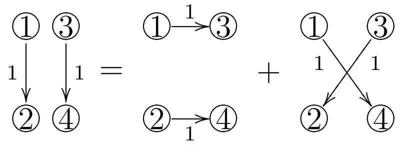
Definition 4.2.8. A node ⓙ is said to be *i*-saturated if there are d_i edges of color i incident to ⓙ.

Remark 4.2.9 (Basic operations). Let G be a graph containing an edge $(1,2)$ of color 1. The following relations hold:

1. If the vertex ③ is not 1-saturated, then $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ becomes



2. If G has an edge $(3,4)$ of color 1, then $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline\end{array}$ becomes



Proposition 4.2.10. *Let λ be as before, and let*

$$e_\lambda = \sum_{i=1}^{n} \lambda_2^i.$$

If $e_\lambda \geq r-1$, then $\mathrm{hwt}_\lambda(U/F)$ is spanned by connected graphs. If $e_\lambda < r-1$, then $\mathrm{hwt}_\lambda(U/F)$ is spanned by graphs G that consist of a tree, together with a collection of isolated nodes.

Proof. We first show that if G has two connected components H_1, H_2 with H_1 containing a cycle, then we can write $G = G_1 + G_2$, where G_1 and G_2 are graphs obtained from G by joining the components H_1, H_2 together.

Consider an edge $(1,2)$ contained in a cycle of H_1, having say color 1. Consider a node ③ of H_2 and suppose first it is not 1-saturated. Using the first basic operation of Remark 4.2.9, we get that $G = G_1 + G_2$, where G_1, G_2 are obtained from G by connecting H_2 to H_1 via an edge of color 1. If ③ is 1-saturated, then in particular there exists at least one edge, say $(3,4)$, of color 1 in H_2. The second basic operation of Remark 4.2.9 yields $G = G_1 + G_2$, where G_1, G_2 are obtained from G by connecting H_1 and H_2 via two edges of color 1.

If $e_\lambda \geq r-1$, then G will contain cycles as long as it is not connected, so iterating the above procedure, we can write G as a linear combination of connected graphs.

If $e_\lambda < r-1$, then the above argument reduces the problem to the case when G is a union of trees, some of which may be isolated nodes. We show that if G has at least two components that are not nodes, then $G = G_1 + G_2$, where G_1, G_2 are unions of trees, and the sizes of the largest components of G_1, G_2 are strictly larger than the size of the largest component of G. Induction on the size of the largest component of G concludes then the proof of the proposition.

Let H_1 be the largest component of G, and let H_2 be another component which isn't a node. If H_2 has only one edge, then $G = 0$ by Corollary 4.2.5. Consider a leaf of H_1, say ③, and assume first that all edges in H_2 have the same color, say 1. Since H_2 has more than one edge and is connected, it must have a vertex with at least two incident edges of color 1, i.e. $d_1 \geq 2$. This means that ③ is not 1-saturated. Let $(1,2)$ be an edge of H_2 (of color 1). The first basic operation of Remark 4.2.9 shows that $G = G_1 + G_2$, where G_1, G_2 are obtained from G by expanding its largest component.

Assume now that the edges in H_2 have at least two colors, and that the edge incident to ③ has color 2. Let $(1,2)$ be an edge of H_2 of color different from 2, say 1. ③ is not 1-saturated, thus we can use the first basic operation of Remark 4.2.9 as in the preceding case. □

4.2.3 Step 3

Combining Step 1 with Step 2 we get that, depending on the n-partition λ, $\mathrm{hwt}_\lambda(U/F)$ is spanned either by connected graphs without odd cycles, or by graphs consisting of a tree and some isolated nodes. We call these graphs maximally connected bipartite (MCB) graphs. For an MCB-graph G, the maximal connected component admits an essentially unique bipartition of its vertex set into subsets A, B of sizes $a \geq b$ (i.e. vertices in the same subset A or B are not connected by an edge). We say that G has type $(a, b; \lambda)$ (or just (a, b) when λ is understood), and that it is canonically oriented if all the edges have source in A and target in B (when $a = b$, there are two canonical orientations). We have the following

Proposition 4.2.11. *If G_1, G_2 are canonically oriented MCB-graphs of type (a, b), then $G_1 = G_2$.*

We first need to refine the relations of Remark 4.2.9:

Remark 4.2.12 (Refined basic operations). Suppose that G is an MCB-graph with vertex bipartition $A \sqcup B$ as above.

1. Assume that ③ is not 1-saturated, $(1, 2)$ is an edge of color 1, and ①, ③ belong to A. If ①, ③ are contained in the same connected component of the graph obtained from G by removing the edge $(1, 2)$, then

$$\begin{tikzcd}[column sep=small] & \textcircled{1} \arrow[dl, "1"'] & & & \textcircled{1} & \\ \textcircled{2} & & \textcircled{3} & = & \textcircled{2} & \textcircled{3} \arrow[l, "1"] \end{tikzcd}$$

   This follows from the fact that the above conditions guarantee that the term that was left out from the first basic operation of Remark 4.2.9 has an odd cycle, and hence equals 0 by Proposition 4.2.2.

2. Assume that $(1, 2)$ and $(3, 4)$ are edges of color 1, ①, ③ $\in A$ and ②, ④ $\in B$, and either ① and ③, or ② and ④ are in the same connected component of the graph obtained from G by removing the edges $(1, 2)$ and $(3, 4)$. Then

$$\begin{tikzcd}[column sep=small] \textcircled{1} \arrow[d, "1"'] & \textcircled{3} \arrow[d, "1"] & & \textcircled{1} \arrow[dr, "1"', near start] & \textcircled{3} \arrow[dl, "1", near start] \\ \textcircled{2} & \textcircled{4} & = & \textcircled{2} & \textcircled{4} \end{tikzcd}$$

   As above, the missing term from the second basic operation has an odd cycle, and hence equals 0.

Proof of Proposition 4.2.11. We prove by induction on e_λ (the number of "edges" of λ), that it is possible to get from G_1 to G_2 via a series of refined basic operations. If $e_\lambda = 0$, there is nothing to prove. Suppose now that $e_\lambda > 0$.

We call an edge E of an MCB-graph G nondisconnecting if the graph obtained from G by removing E is still an MCB-graph. More explicitly, if $e_\lambda \geq r$, then E must be contained in a cycle of G, and if $e_\lambda < r$, then one of the endpoints of E must be a leaf of G.

We will prove that for any nondisconnecting edge E_2 of G_2 of color c, there exist a sequence of refined basic operations which transforms G_1 into a new graph $\hat{G}_1$ having a nondisconnecting edge E_1 of color c, such that the graphs G'_1 and G'_2 obtained from $\hat{G}_1$ and G_2 by removing the edges E_1 and E_2 have the same type. Assuming this, by induction we can find a series of refined basic operations that transform G'_1 into G'_2. We lift this sequence of operations to $\hat{G}_1$ as follows: the refined basic operations of type (2) are performed just as if the edge E_1 was not contained in $\hat{G}_1$, as well as the operations of type (1) that don't transform an edge E' of color c into one that's incident to E_1; the operations of type (1) involving an edge E' of color c that gets transformed into an edge incident to E_1 are replaced by operations of type (2) involving E' and E_1. It is clear that E_1 remains nondisconnecting along the process, so we end up with the graphs G_1'' and G_2 that coincide after removing the nondisconnecting edges E_1 and E_2 of color c. At most two more refined operations of type (2) (that correspond to correcting the positions of the endpoints of E_1) are then sufficient to transform G_1'' into G_2, concluding the proof.

We now show that if $e_\lambda \geq r$ and G_1 has an edge E_1 of color c, then we can find a refined basic operation that makes E_1 nondisconnecting. Suppose that E_1 is disconnecting, and let H_1, H_2 be the connected components of the graph obtained from G_1 by removing the edge E_1. One of H_1, H_2 must contain a cycle, say H_1, and let O, Y be consecutive edges of this cycle, of colors $o(range)$ and $y(ellow)$ (note that o might coincide with y). If H_2 has a node N that is not o-saturated or not y-saturated, then a refined operation of type (1) involving the node N (as ③) and one of the edges O, Y (as the edge $(1,2)$) will make E_1 a nondisconnecting edge. Otherwise, if every vertex of H_2 is both o- and y-saturated, then there exists a cycle in H_2 consisting of edges of colors o and y (if $o = y$, then since O, Y are incident edges of color o, it means that $d_o \geq 2$, in particular any o-saturated node has at least two incident edges of color o; if $o \neq y$, then any o- and y-saturated node has at least one o- and one y- incident edge; in both cases, the nodes in H_2 have at least two incident edges, so we can find a cycle as stated). A refined basic operation of type (2) involving an o-edge on this cycle and O (or an y-edge and Y) will make E_1 nondisconnecting.

Finally, if $e_\lambda < r$ and G_2 has a nondisconnecting edge E_2 of color c, then we have claimed that we can find a sequence of refined basic operations that transforms G_1 into a graph $\hat{G}_1$ containing a nondisconnecting edge E_1 of color c, and moreover $\hat{G}_1 - E_1$ and $G_2 - E_2$ have the same type. We may assume that $e_\lambda = r - 1$, by removing the isolated nodes of G_1 and G_2. Suppose that the graphs G_i have vertex bipartitions $A_i \sqcup B_i$, with $|A_i| = a$, $|B_i| = b$, and that $E_2 = (x, y)$, with ⓨ $\in B_2$ a leaf of G_2. This means that the graph G_2, and hence also G_1, has at most $(b-1) \cdot d_c + 1$ edges of color c, and for any color $c' \neq c$, it has at most $(b-1) \cdot d_{c'}$ edges of color c'. In particular, for any color $c' \neq c$, there exists a node in B_1 which is not c'-saturated. Consider an edge $E = (u, v)$ of color c in G_1, with ⓤ $\in A_1$, ⓥ $\in B_1$. Let H_1, H_2 be the connected components of $G_1 - E$ containing u and v respectively. We prove by descending induction on the size of H_2 that we can make E nondisconnecting, with its endpoint in B_1 being a leaf.

If $H_2 = \{ⓥ\}$ then E is nondisconnecting. More generally, if $H_2 \cap B_1 = \{ⓥ\}$, then we may assume that all the edges in H_2 have color c. If E' is an edge of H_2 of color $c' \neq c$ (see the second transformation in Example 4.2.13 below), then there are at most $(b-1) \cdot d_{c'} - 1$ edges of color c' in H_1, so that we can find a vertex C' in H_1 that is not c'-saturated. A refined basic operation of type (1) involving E' and C' decreases the size of H_2 by one, so we can conclude by induction. Assume now that the edges in H_2 have color c. Together with the edge E, we get at least two edges of color c outside H_1, which means that H_1 has at most $(b-1) \cdot d_c - 1$ edges of color c, i.e. it has a vertex that is not c-saturated. We now do a refined basic operation of type (1) as before, involving that vertex and an edge of H_2, and conclude by induction.

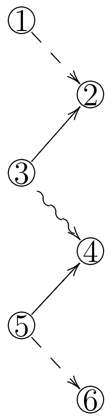
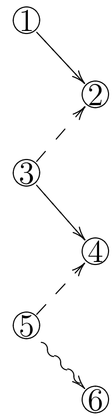
We may now assume that $|H_2 \cap B_1| > 1$ (see the first transformation in Example 4.2.13

below). Therefore there exist distinct edges $Y = (u', v)$ of color $y(ellow)$ and $O = (u', v')$ of color $o(range)$ in H_2 (y and o might coincide). If $o = c$ then we replace E with (u', v'), which decreases the size of H_2, so that we can conclude by induction. If there exists a vertex $W \in H_1 \cap B_1$ that is not y-saturated, then the refined basic operation involving Y and W decreases the size of H_2. Likewise, if there exists a vertex $W \in H_1 \cap A_1$ that is not o-saturated, then the refined basic operation involving O and W also decreases the size of H_2. We may therefore assume that all nodes in $B_1 \cap H_1$ are y-saturated, and those in $A_1 \cap H_1$ are o-saturated, and show that this leads to a contradiction. If $y = o$, then since u' has two incident edges of color o, we must have $d_o \geq 2$. All the nodes of H_1 being saturated implies that they have degree at least $d_o \geq 2$, so H_1 contains a cycle, which is a contradiction. If $y \neq o$, then H_1 must contain at least $|H_1 \cap A_1|$ edges of color o (since each vertex in $H_1 \cap A_1$ is o-saturated) and at least $|H_1 \cap B_1|$ edges of color y, i.e. H_1 contains at least $|H_1|$ edges, hence it can't be a tree. $\square$

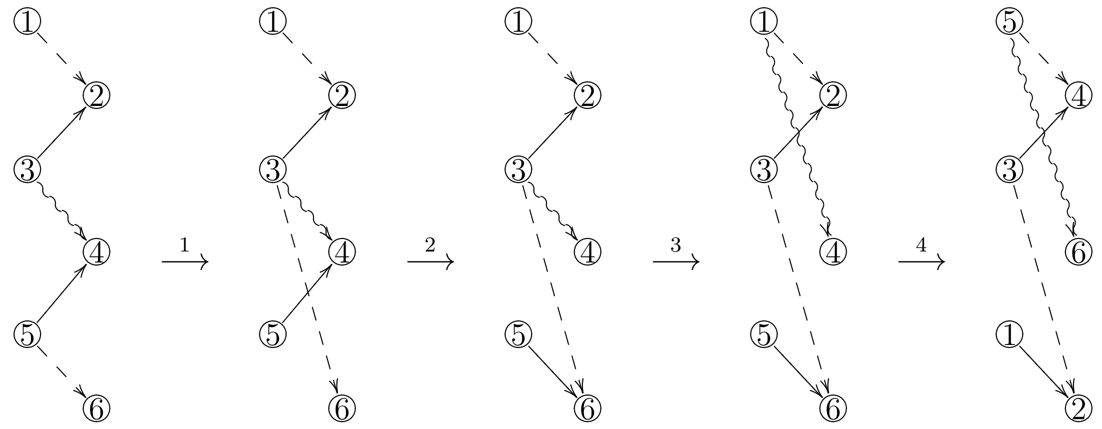
Example 4.2.13. Consider the 3-tableaux

$$T_1 = \begin{array}{|c|c|c|c|}\hline 3&5&1&6\\ \hline 2&4 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 3&1&2&5&6\\ \hline 4 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&5&3&4\\ \hline 2&6 \\ \cline{1-2}\end{array}, \quad T_2 = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&1&2&3&4\\ \hline 6 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 3&5&1&6\\ \hline 2&4 \\ \cline{1-2}\end{array},$$

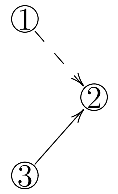
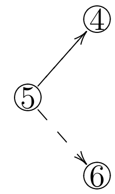
with corresponding graphs

G_1 =  and G_2 = 

where color 1 corresponds to $\longrightarrow$, color 2 to $\rightsquigarrow$, and color 3 to $\dashrightarrow$. G_1 and G_2 are MCB of the same type, and in fact $G_1 = 0$, since it is the same as the graph obtained by reversing the orientation of its 5 edges (an odd number), and this equals $-G_1$ by part a) of Lemma 4.2.3. However, it is unclear a priori that G_2 is also equal to 0. We use the algorithm described in the proof of Proposition 4.2.11 to get a sequence of refined basic operations that transforms G_1 into G_2. We first make the edge of G_1 of color 2 nondisconnecting, and then adjust its position (the third step) and relabel the nodes (last step) to get G_2:

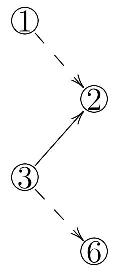
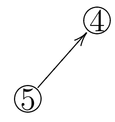


With the notation in the last paragraph of the proof of Proposition 4.2.11, we have $E = (3, 4)$ a disconnecting edge, $A_1 = \{1, 3, 5\}$, $B_1 = \{2, 4, 6\}$ a bipartition of the vertex set of G_1. We'd like to make E nondisconnecting, with its endpoint in B_1 being a leaf. We have

$H_1 \quad =$  and $H_2 \quad =$ 

We also have $Y = (5, 4)$ of color $y = \longrightarrow$ and $O = (5, 6)$ of color $o = \dashrightarrow$. The unique vertex ② in $H_1 \cap B_1$ is y-saturated, and ① $\in H_1 \cap A_1$ is o-saturated, but $W =$ ③ is not o-saturated. The refined basic operation involving W and O yields the first transformation.

We now have

$H_1 \quad =$  and $H_2 \quad =$ 

We are in the case $H_2 \cap B_1 = \{ⓥ\} = \{④\}$. The edge $E' = (5, 4)$ has color $c' = \longrightarrow$, different from $c = \rightsquigarrow$. $W =$ ⑥ is a vertex in $H_1 \cap B_1$ which is not c'-saturated, so we can use the refined basic operation involving E' and W as our second transformation, making E a nondisconnecting edge as desired.

We next adjust the position of E, in order to get the graph G_2. We use the refined operation involving the vertex ① and the edge $(3, 4)$. The last transformation involves relabeling the nodes ⑤, ⑥, ②, ① and ④ by ①, ②, ④, ⑤ and ⑥ respectively.

Corollary 4.2.14. *If G is a canonically oriented MCB-graph of type (a, a), having an odd number of edges, then $G = 0$.*

Proof. Changing the orientation of all the edges of G, we obtain a canonically oriented MCB-graph G' of the same type as G. It follows from Proposition 4.2.11 that $G = G'$. On the other hand, we get by part a) of Lemma 4.2.3 that $G' = -G$, hence $G = 0$. $\square$

4.2.4 Step 4

The preceding steps yield the following

Corollary 4.2.15. *For e_λ, f_λ as in Theorem 4.1.1, the space $(U/F)_\lambda$ is spanned by MCB-graphs $G_{\mu'}$ of type $\mu' = (a' \geq b')$, with $a' + b' = \min(e_\lambda + 1, r)$ and $b' \geq f_\lambda$. Moreover, $G_{\mu'} = 0$ if $a' = b'$ and e_λ is odd.*

Proof. The last statement is the content of Corollary 4.2.14. We know that $(U/F)_\lambda$ is spanned by MCB-graphs (Proposition 4.2.10), and the condition $b' \geq f_\lambda$ follows from the fact that any graph G has at least λ_2^i/d_i vertices incident to edges of color i, and any edge is incident to one vertex in each of the two sets of the bipartition. The number of vertices in the maximal connected component of an MCB-graph of type μ' is $a' + b' = \min(e_\lambda + 1, r)$.

It remains to show that if $\mu' = (a' \geq b')$, $a' + b' = \min(e_\lambda + 1, r)$ and $b' \geq f_\lambda$, then there exists an MCB-graph $G_{\mu'}$ of type μ'. Consider A' and B' disjoint sets consisting of a' and b' vertices in $\{①, \cdots, ⓡ\}$ respectively. For every $i = 1, \cdots, n$ we draw λ_2^i edges of color i joining pairs of elements in A' and B', in such a way that no vertex has more than d_i incident edges of color i. This is possible since $\lambda_2^i/d_i \leq f_\lambda \leq b' \leq a'$. If the bipartite graph G (with vertex set $A' \cup B'$) obtained in this way is connected, then we get an MCB-graph $G_{\mu'}$ by adding to G the isolated nodes outside $A' \cup B'$. If G is not connected, then it has an edge E of color c contained in a cycle, and a vertex v outside the connected component of E. If v is not c-saturated, we can move E to make it incident to v, and preserve the bipartition of G (as in the refined basic operations of type (1), Remark 4.2.12), thus obtaining a graph with fewer components. If v is c-saturated, let E' be an incident edge of color c. We move E and E' as in a refined basic operation of type (2), connecting the components of E and v. Repeating this procedure will eventually yield a connected graph G and an MCB-graph $G_{\mu'}$ as above. $\square$

Lemma 4.2.16. *Consider canonically oriented graphs $G_{\mu'}$ as above, one for each type $\mu' = (a', b')$, with $a' \neq b'$ when e_λ is odd. If*

$$\pi = \bigoplus_{\substack{\mu \vdash r \\ \mu = (a \geq b)}} \pi_\mu : U \longrightarrow \bigoplus_{\substack{\mu \vdash r \\ \mu = (a \geq b)}} U_\mu^d,$$

then the set $\{\pi(G_{\mu'})\}_{\mu'}$ is linearly independent. In particular, $F = I$ and the graphs $G_{\mu'}$ give a basis of $(U/F)_\lambda$. This shows that $\dim((U/I)_\lambda) = m_\lambda$*, where m_λ is as described in Theorem 4.1.1, concluding the proof of our main result.*

Proof. Note that the number of $G_{\mu'}$'s is precisely m_λ, so the last statement follows once we prove the independence of $G_{\mu'}$'s. This is a consequence of the linear independence of $\{\pi(G_{\mu'})\}_{\mu'}$, which in turn follows once we show that for $\mu = (a, b)$, $\mu' = (a', b')$, we have

1. $\pi_\mu(G_{\mu'}) = 0$ if $b < b'$, and
2. $\pi_\mu(G_{\mu'}) \neq 0$ if $b = b'$.

Recall that $G_{\mu'} = T_{\mu'}$, for some n-tableau $T_{\mu'}$. We have

$$\pi_\mu(T_{\mu'}) = \sum T_i, \tag{*}$$

where each T_i is an n-tableau with entries $1, 2$, obtained from a partition $A \sqcup B = \{1, \cdots, r\}$, by setting equal to 1 and 2 the entries of $T_{\mu'}$ from A and B respectively.

To prove (1), note that since $|B| = b < b'$, for each i the endpoints of some edge in $G_{\mu'}$ have to be set to the same value, so T_i has repeated entries in some column, i.e. $T_i = 0$. It follows that $\pi_\mu(G_{\mu'}) = \sum T_i = 0$.

To prove (2), let $A' \sqcup B'$ be the bipartition of the maximal connected component of $G_{\mu'}$, and take $\mu = (a, b) = (d - b', b')$. The only n-tableau(x) T_i in (*) without repeated entries in some column is (are) the n-tableau T_1 obtained from setting the entries of $A = \{1, \cdots, r\} - B'$ to 1, and the entries of $B = B'$ to 2 (and if $|A'| = |B'|$, the n-tableau T_2 obtained by setting the entries of $A = \{1, \cdots, r\} - A'$ to 1 and the entries of $B = A'$ to 2). Since in the latter case e_λ must be even, we get in fact that $T_1 = T_2$, since T_1 and T_2 differ by an even number of transpositions within columns, and by permutations of columns of size 1. It follows that it's enough to prove that $T_1 \neq 0$.

Up to permutations within columns, and permutations of columns of the same size, we may assume that

$$T_1 = c_\lambda \cdot m = c_\lambda \cdot z_{(A,\cdots,A)} \cdot z_{(B,\cdots,B)},$$

where $A = \{1, \cdots, a\}$ and $B = \{a + 1, \cdots, a + b\}$, i.e. $T_1 = T_1^1 \otimes \cdots \otimes T_1^n$, with

$$T_1^i = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 2 & 2 & \cdots \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4}\end{array}.$$

If $a > b$ and $\sigma = \tau \cdot \tau'$, with τ a row permutation and τ' a column permutation of the canonical n-tableau T_λ of shape λ, then $\sigma \cdot m \neq m$, unless $\tau' = id$. This shows that the coefficient of m in T_1 is a positive number, hence $T_1 \neq 0$. If $a = b$, $\sigma \cdot m = m$ and $\tau' \neq id$, then τ' must transpose all the pairs $(1, 2)$ in the columns of T_1 of size 2. Since T_1 has e_λ (an even number) of such columns, the signature of τ' must be $+1$. It follows again that the coefficient of m in T_1 is positive and therefore $T_1 \neq 0$. □

□

Chapter 5

Examples

The purpose of this chapter is to gather some known examples from the literature, and describe them from the "generic" perspective. The list of examples we consider consists of

- Cauchy's formula for decomposing a symmetric power of a tensor product of two vector spaces.
- The (generalized) Strassen's equations, which are modules of equations for secant varieties of Segre varieties.
- Equations of degree 6 of the third secant variety of a triple Segre product of projective 3-spaces.
- The Aronhold invariant, a module of equations for the variety of secant planes to the 3-uple embedding of $\mathbb{P}^2$.

The only new contribution of this chapter is showing that the generalized Strassen's equations occur in the generality anticipated by the authors of [LM08]. More precisely, we relax the condition $r \geq 3s$ from [LM08, Theorem 4.2] to $r \geq 2s$.

5.1 Tableaux

Let $\underline{r} = (r_1, \cdots, r_n)$ be a sequence of positive integers, $\lambda = (\lambda^1, \cdots, \lambda^n)$ an n-partition of $\underline{r}$, r a positive integer giving rise to an alphabet $\mathcal{A} = \{1, \cdots, r\}$, and $\underline{n} = (n_i^j)_{\substack{i=1,\cdots,r\\ j=1,\cdots,n}}$ an array of nonnegative integers satisfying

$$\sum_{i=1}^{r} n_i^j = r_j, \quad j = 1, \cdots, n.$$

Definition 5.1.1. We define $\mathrm{Tab}(\lambda, \underline{r}, \underline{n})$ to be an (the) universal vector space of n-tableaux $T = T^1 \otimes \cdots \otimes T^n$ with T^j containing n_i^j entries equal to i, and satisfying the following relations

1. *(skew-symmetry on columns)* A permutation σ of the entries within a column of some tableau T^j in T multiplies T by $\mathrm{sgn}(\sigma)$, the sign of the permutation σ.

2. *(shuffling relations)* Given two columns C_1, C_2 of some tableau T^j of sizes $|C_1| \geq |C_2|$, and given subsets $B_i \subset C_i$ of their sets of boxes, with the property that $|B_1| + |B_2| > |C_1|$, then we have the shuffling relation
$$\sum_{\sigma \in S_B} \mathrm{sgn}(\sigma) \cdot \sigma(T) = 0,$$
where $B = B_1 \cup B_2$, S_B denotes the group of permutations of the boxes in B, and $\sigma \in S_B$ acts on an n-tableau T by permuting the selected set B of boxes.

3. *(symmetry)* If $i \neq i' \in \mathcal{A}$ have the property that $n_i^j = n_{i'}^j$ for all $j = 1, \cdots, n$, then $T = T_{i \leftrightarrow i'}$ for any n-tableau T, where $T_{i \leftrightarrow i'}$ is the n-tableau obtained from T by interchanging i with i'.

Remark 5.1.2. The reason we have stated the *shuffling relations* in this form is in order to stick to the standard literature ([Wey03, Chapter 2]). We will only use them in the form that appears in part (3) of Lemma 3.2.10.

Example 5.1.3. Let $\underline{r} = (rd_1, \cdots, rd_n)$, and $n_i^j = d_j$ for all i, j. Then $\mathrm{Tab}(\lambda, \underline{r}, \underline{n})$ can be identified with the λ-highest weight space of the representation $U_{\underline{r}}^{\underline{d}}$ introduced in Definition 3.2.1. $\mathrm{Tab}(\lambda, \underline{r}, \underline{n}) = c_\lambda \cdot U_{\underline{r}}^{\underline{d}}$ is a vector space of dimension equal to the multiplicity of the irreducible $GL(V)$-representation $S_\lambda V$ inside $S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)$ (when the dimensions of the V_i's are large enough so that $S_\lambda V \neq 0$). Alternatively, its dimension equals the multiplicity of the irreducible $S_{\underline{r}}$-representation $[\lambda]$ inside the induced representation $\mathrm{Ind}^{S_{\underline{r}}}_{(S_{d_1} \times \cdots \times S_{d_n})^r \wr S_r}(\mathbf{1})$.

Example 5.1.4. This example's goal is to illustrate the three parts of Definition 5.1.1.

1. We have
$$\ytableaushort{12,3} \otimes \ytableaushort{1,2,3} \otimes \ytableaushort{112,2} = -\ytableaushort{12,3} \otimes \ytableaushort{1,3,2} \otimes \ytableaushort{112,2},$$
because the 3-tableau on the right hand side was obtained from the one on the left by making a transposition $(2,3)$ of two boxes in its middle tableau.

   Now change all the 3's in the previous example to 2's. By the same argument, we get
$$T = \ytableaushort{12,2} \otimes \ytableaushort{1,2,2} \otimes \ytableaushort{112,2} = -\ytableaushort{12,2} \otimes \ytableaushort{1,2,2} \otimes \ytableaushort{112,2} = -T,$$

hence $T = 0$. In general, tableaux with repeated entries in some column are equal to zero.

2. For any tableau T^2, we have the following relation between 2-tableaux:

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \otimes T^2 = \begin{ytableau} 2 & 1 \\ 3 \end{ytableau} \otimes T^2 + \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \otimes T^2.$$

(see part (3) of Lemma 3.2.10 for the more general statement; see also part (2) of Lemma 4.2.3 for this specific situation.)

3. The equality below already showed up in Example 3.2.9. It comes from applying the permutation $(1, 3, 4, 2)$ (in cycle notation) to the entries of the alphabet $\mathcal{A} = \{1, 2, 3, 4\}$.

$$\begin{ytableau} 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 4 \\ 2 \end{ytableau} = \begin{ytableau} 3 & 1 & 1 & 4 & 4 \\ 3 & 2 & 2 \end{ytableau} \otimes \begin{ytableau} 3 & 4 \\ 2 \\ 1 \end{ytableau}.$$

5.2 Cauchy's Formula

Cauchy's formula is a plethystic formula for decomposing a symmetric power of a tensor product of two vector spaces into a sum of irreducible representations over the product of the corresponding general linear groups. More precisely, it says the following.

Proposition 5.2.1 (Cauchy's formula)**.** *Let V_1, V_2 be vector spaces over a field of characteristic zero, and let r be a positive integer. We have*

$$S_{(r)}(V_1 \otimes V_2) = \bigoplus_{\mu \vdash r} S_\mu V_1 \otimes S_\mu V_2.$$

Proof. We are in the situation of Example 5.1.3, with $n = 2$ and $d_1 = d_2 = 1$. The statement of Cauchy's formula is equivalent to the fact that the vector space $\mathrm{Tab}(\lambda, \underline{r}, \underline{n})$ has dimension one if $\lambda^1 = \lambda^2 = \mu \vdash r$ and is zero otherwise. Fix then a pair of partitions $\lambda^1, \lambda^2 \vdash r$, and let $T = T^1 \otimes T^2 \in \mathrm{Tab}(\lambda, \underline{r}, \underline{n})$ be a tableau.

By property 3. of Definition 5.1.1, we may assume that the entries in T^1 are ordered top to bottom and left to right, as in

$$\begin{ytableau} 1 & 4 & 6 & 7 \\ 2 & 5 \\ 3 \end{ytableau}.$$

We will show that we can assume the same for the entries of T^2, and moreover, that if T^1 and T^2 don't have the same shape, then in fact $T = 0$. We prove this column by column, proceeding from left to right. We may assume that T^1 has at least as many boxes (say c) in the first column as T^2 does. Suppose that we've arranged that T^2 has the first i entries in its first column equal to $1, 2, \cdots, i$ respectively, for some $0 \le i \le c$. If $i = c$, then the first columns of T^1 and T^2 coincide, and we can proceed to the next column.

Suppose now that $i < c$. Let's assume first that T^2 has exactly i boxes in its first column. We write down the shuffling relation with B_1 being the set of boxes in the first column of T^2, and $B_2 = \{\boxed{i+1}\}$ consisting of the unique box in T^2 having its entry equal to $i+1$. We get

$$T^1 \otimes \begin{ytableau} 1 & i{+}1 \\ 2 \\ \vdots \\ i \end{ytableau} = T^1 \otimes \begin{ytableau} i{+}1 & 1 \\ 2 \\ \vdots \\ i \end{ytableau} + T^1 \otimes \begin{ytableau} 1 & 2 \\ i{+}1 \\ \vdots \\ i \end{ytableau} + \cdots + T^1 \otimes \begin{ytableau} 1 & i \\ 2 \\ \vdots \\ i{+}1 \end{ytableau}$$

Now interchanging the entries equal to j with those equal to $i+1$ in the j-th term on the RHS of the shuffling relation above preserves the corresponding 2-tableau by symmetry (property 3. of Definition 5.1.1). This has the effect of applying a transposition to the first column of T^1, and making the 2-nd tableau of the j-th term equal to T^2. Using the skew-symmetry of tableaux on columns to cancel the effect of the transposition on T^1, we obtain

$$T = -T - T - \cdots - T = -i \cdot T,$$

hence $(i+1) \cdot T = 0$, yielding $T = 0$. Here's an example to illustrate the argument of the preceding paragraph:

Example 5.2.2. Let $T^1 = \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau}$, $T^2 = \begin{ytableau} 1 & 4 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau}$, $T = T^1 \otimes T^2$, $c = 4$ and $i = 3$. Using the shuffling relation involving the boxes with entries $\{1, 2, 3, 4\}$ in T^2, we get

$$T = \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1 & 4 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 4 & 1 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1 & 2 & 5 \\ 4 & 7 & 6 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1 & 3 & 5 \\ 2 & 7 & 6 \\ 4 \end{ytableau}.$$

Now applying the transpositions $(1, 4)$, $(2, 4)$ and $(3, 4)$ respectively to the three terms on the RHS of the above relation, we get

$$T = \begin{ytableau} 4 & 5 & 7 \\ 2 & 6 \\ 3 \\ 1 \end{ytableau} \otimes \begin{ytableau} 1 & 4 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 5 & 7 \\ 4 & 6 \\ 3 \\ 2 \end{ytableau} \otimes \begin{ytableau} 1 & 4 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 5 & 7 \\ 2 & 6 \\ 4 \\ 3 \end{ytableau} \otimes \begin{ytableau} 1 & 4 & 5 \\ 2 & 7 & 6 \\ 3 \end{ytableau}.$$

Note that the 2-nd tableau in each term is now equal to T^2, and that the first tableau is obtained from T^1 by applying a transposition. Since tableaux are skew-symmetric on columns, it follows that each of the terms is $-T^1 \otimes T^2 = -T$, i.e. $T = -3T$, yielding $T = 0$.

Assume now that T^2 has more than i boxes in its first column. Then the same argument as above shows that $(i+1)\cdot T$ is a linear combination of 2-tableaux of the form $T^1 \otimes T'$, where T' is a tableau whose first column contains the entries $\{1, 2, \cdots, i+1\}$, so we can conclude by induction on i. Below is an example to illustrate this phenomenon:

Example 5.2.3. Let $T^1 = \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$, $T^2 = \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array}$, $T = T^1 \otimes T^2$, $c = 4$ and $i = 3$. Using the shuffling relation involving the boxes with entries $\{1, 2, 3, 6, 4\}$ in T^2, we get

$$T = \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 4&1&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&7\\\cline{1-2} 4\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&6&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}.$$

Now applying the transpositions $(1,4)$, $(2,4)$ and $(3,4)$ respectively to the first three terms on the RHS of the above relation, we get

$$T = \begin{array}{|c|c|c|}\hline 4&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 1\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 4&6\\\cline{1-2} 3\\\cline{1-1} 2\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1} 3\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&6&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}.$$

Note that the 2-nd tableau in each of the first three terms on the RHS is now equal to T^2, and that the first tableau is obtained from T^1 by applying a transposition. Since tableaux are skew-symmetric on columns, it follows that each of the terms is $-T^1 \otimes T^2 = -T$, i.e.

$$T = -3T + \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&6&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array},$$

yielding

$$T = \frac{1}{4} \cdot \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&6&5\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}.$$

Repeating the arguments above to the other columns of T^1 and T^2 we obtain that $\mathrm{Tab}(\lambda, \underline{r}, \underline{n}) = 0$ when $\lambda^1 \neq \lambda^2$ and is at most 1-dimensional when $\lambda^1 = \lambda^2 = \mu$, generated by $T^1 \otimes T^1$ where T^1 is the tableau of shape μ with entries ordered top to bottom and left to right (or any other tableau of shape μ, by symmetry). To see that $T = T^1 \otimes T^1$ is nonzero, note that

$$T = c_\lambda \cdot m, \text{ where } m = \prod_{i=1}^{r} z_{(\{i\},\{i\})}.$$

We can write

$$c_\lambda \cdot m = \sum_{\sigma\in R_\lambda,\ \tau\in C_\lambda} \mathrm{sgn}(\tau)\cdot\sigma\cdot\tau\cdot m,$$

where R_λ and C_λ (Section 2.3) denote the groups of row and column permutations respectively. We write $\sigma = (\sigma_1, \sigma_2)$ and $\tau = (\tau_1, \tau_2)$. In order for m and $\sigma\cdot\tau\cdot m$ to be equal, we must have that $\sigma_1\tau_1 = \sigma_2\tau_2$. Since R_λ and C_λ intersect trivially, we must have $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$, in which case $\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau_1)\mathrm{sgn}(\tau_2) = 1$. It follows that all the occurrences of m in $c_\lambda \cdot m$ have coefficient 1, thus $T \neq 0$. $\square$

5.3 Strassen's Equations

We give the generic version of Strassen's equations ([Str88]) for $\sigma_3(\mathrm{Seg}(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2))$. These equations have been shown by Landsberg and Weyman to generate the homogeneous ideal of $\sigma_3(\mathrm{Seg}(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2))$ ([LW07]).

We are in the situation of Example 5.1.3, with $n = 3$, $r = 4$, $d_1 = d_2 = d_3 = 1$ and $\lambda^1 = \lambda^2 = \lambda^3 = (2,1,1) = (1^2 2)$. We show that $\mathrm{Tab}(\lambda, \underline{d}, \underline{n})$ has a basis consisting of the 3-tableau

$$T_0 = \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau},$$

and that T_0 gives a generic equation for the variety of secant planes to a triple Segre product of projective spaces. That is, we show that for every partition $\mu \vdash 4$ with three parts, we have $\pi_\mu(T_0) = 0$, where π_μ is as defined in 3.2.5.

To prove the last statement, note that $\mu = (2,1,1)$ is the only partition of 4 with three parts. Note also that $\pi_\mu(T_0)$ is in this case a sum of 3-tableaux obtained from T_0 by setting two of the entries $\{1,2,3,4\}$ equal to 1 and the other equal to 2 and 3 in some (any) order. In particular, $\pi_\mu(T_0)$ is a sum of 3-tableaux with repeated entries in some column (because every pair $\{i,j\} \subset \{1,2,3,4\}$ shows up in at least one column of T_0), hence $\pi_\mu(T_0) = 0$. Explicitly,

$$\pi_\mu(T_0) = \begin{ytableau} 1 & 1 \\ 3 \\ 2 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 1 \\ 3 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 1 \\ 2 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 1 \\ 3 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 2 \\ 1 \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 \\ 1 \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 \\ 3 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 2 \\ 1 \end{ytableau}$$

$$+ \begin{ytableau} 2 & 1 \\ 1 \\ 3 \end{ytableau} \otimes \begin{ytableau} 2 & 3 \\ 1 \\ 1 \end{ytableau} \otimes \begin{ytableau} 2 & 1 \\ 1 \\ 3 \end{ytableau} + \begin{ytableau} 2 & 1 \\ 3 \\ 1 \end{ytableau} \otimes \begin{ytableau} 2 & 1 \\ 1 \\ 3 \end{ytableau} \otimes \begin{ytableau} 2 & 3 \\ 1 \\ 1 \end{ytableau} + \begin{ytableau} 3 & 2 \\ 1 \\ 1 \end{ytableau} \otimes \begin{ytableau} 3 & 1 \\ 2 \\ 1 \end{ytableau} \otimes \begin{ytableau} 3 & 1 \\ 2 \\ 1 \end{ytableau},$$

so we see that our claim is indeed true.

To see that T_0 spans, it suffices to note that if two tableaux T^i, T^j of a 3-tableau T of shape λ have the same entry in the 2-nd column, then $T = 0$. Say the two tableaux are

T^1, T^2 and 4 is their entry in the second column. Up to sign (because of the skew-symmetry of tableaux), we may assume that

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes T^3.$$

Then no matter what T^3 is, one of the pairs $\{1,2\}, \{1,3\}, \{2,3\}$ shows up in the first column of each of the T^i's. Interchanging the two entries in that pair simultaneously preserves T (by symmetry) and changes T to $(-1)^3 T$ (by skew-symmetry), proving that $T = 0$. We may then assume that the tableaux of T have distinct entries in their unique box in the 2-nd column, hence by symmetry we can assume these entries are 2 for T^1, 4 for T^2 and 3 for T^3. We can then rearrange the entries in the first columns of the tableaux of T (at the expense of possibly changing the sign of T) to get T_0.

To see that $T_0 \neq 0$, note that

$$T_0 = c_\lambda \cdot m, \text{ where } m = z_{(\{1\},\{1\},\{1\})} \cdot z_{(\{2\},\{3\},\{3\})} \cdot z_{(\{3\},\{4\},\{2\})} \cdot z_{(\{4\},\{2\},\{4\})},$$

and that each time m appears in $c_\lambda \cdot m$, it has coefficient 1. To see this, note that the instances of m as a term in $c_\lambda \cdot m$ coincide with the ways of obtaining $\sigma(T_0)$ from T_0 for some permutation σ of the entries $\{1,2,3,4\}$, by applying first column permutations and then row permutations to T_0. The coefficient of m corresponding to that instance would then be the product of the signs of the column permutations involved.

Now in order to get $\sigma(T_0)$ from T_0 by applying column permutations followed by row permutations, it must be the case that either $\sigma(1) = 1$, in which case σ is the identity, or that for some $i \in \{2,3,4\}$ we have $\sigma(1) = i$, and $\sigma(j) = j$ for $i \neq j \in \{2,3,4\}$: this is because any of the entries $\{2,3,4\}$ in the 2-nd columns of the T_0^i's either stays fixed, or takes the place of 1. This means that σ is one of the transpositions $(1,2), (1,3), (1,4)$. To realize such a σ precisely two column transpositions of T are required, so the product of the corresponding signs is $+1$. This shows that the coefficient of m in $c_\lambda \cdot m$ is in fact 4, and in any case $T_0 \neq 0$.

5.4 Generalized Strassen's Equations

We give the generic version of the generalized Strassen's equations, or coercive contractions, as they appear in Theorem 4.2 of [LM08]. We also show that this theorem can be strengthen in the case when r is odd, by replacing the condition $r \geq 3s$ with $r \geq 2s$, as anticipated by the authors.

We are in the situation of Example 5.1.3, with $n = 3$, $d_1 = d_2 = d_3 = 1$, and r, the size of the alphabet $\mathcal{A}$ replaced by $r + s$, where r, s are positive integers, s is odd and $r \geq 2s$. We also have $\lambda^1 = (r - s, s, s)$ and $\lambda^2 = \lambda^3 = (2^s 1^{r-s})$. For convenience, we use special notation for some of the elements in the alphabet $\mathcal{A}$: we denote $r-s+i$ by i' and $r+i$ by i'', for $i=1,\cdots,s$; we also denote $r-s$ by $\hat{r}$. The alphabet $\mathcal{A}$ is thus the union of $\{1,\cdots,\hat{r}\}$, $\{1',\cdots,s'\}$ and $\{1'',\cdots,s''\}$.

Proposition 5.4.1. *With the above notation, the vector space* $\mathrm{Tab}(\lambda,\underline{d},\underline{n})$ *is one-dimensional, spanned by the tableau*

$$T_0=\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r}\\ \hline 1' & 2' & \cdots & s' \\ \cline{1-4} 1'' & 2'' & \cdots & s'' \\ \cline{1-4}\end{array}\otimes\begin{array}{|c|c|}\hline 1 & 1''\\ \hline 2 & 2''\\ \hline \vdots & \vdots\\ \hline s & s''\\ \hline \vdots \\ \cline{1-1} \hat{r} \\ \cline{1-1} 1' \\ \cline{1-1} \vdots \\ \cline{1-1} s' \\ \cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1 & 1'\\ \hline 2 & 2'\\ \hline \vdots & \vdots\\ \hline s & s'\\ \hline \vdots \\ \cline{1-1} \hat{r} \\ \cline{1-1} 1'' \\ \cline{1-1} \vdots \\ \cline{1-1} s'' \\ \cline{1-1}\end{array}.$$

Moreover, T_0 *gives a generic equation for the variety of secant* $(r-1)$*-planes to a triple Segre product, i.e. for any partition* $\mu\vdash(r+s)$ *with exactly* r *parts,* $\pi_\mu(T_0)=0$.

Proof. Let us first prove that T_0 spans. Consider any 3-tableau $T\in\mathrm{Tab}(\lambda,\underline{d},\underline{n})$, and let C_2,C_3 be the sets of entries in the first columns of T^2 and T^3 respectively. C_2,C_3 are subsets of size r of the alphabet $\mathcal{A}$ (which has cardinality $r+s$), hence their intersection contains at least $2\cdot r-(r+s)=\hat{r}$ elements. If $|C_2\cap C_3|>\hat{r}$, then since T^1 has only $\hat{r}$ columns, we can find $i\neq j\in C_2\cap C_3$ such that i and j are contained in the same column of T^1. Now the usual argument based on combining properties 1. and 3. of Definition 5.1.1 shows that $T=0$: interchanging the entries i and j in T simultaneously preserves T and changes it to $(-1)^3T$, hence $T=-T$, i.e. $T=0$.

We may therefore assume that $|C_2\cap C_3|=\hat{r}$, and by symmetry that in fact $C_2\cap C_3=\{1,\cdots,\hat{r}\}$. We may further assume still using property 3. of Definition 5.1.1 that $C_2\setminus C_3=\{1',\cdots,s'\}$ and $C_3\setminus C_2=\{1'',\cdots,s''\}$. Using skew-symmetry, we can rearrange (up to sign) the entries in T^2 and T^3 and assume that $T^2=T_0^2$ and $T^3=T_0^3$. The same argument as at the end of the previous paragraph shows that, unless $T=0$, no two elements of $C_2\cap C_3$ can be contained in the same column of T^1, i.e. there is precisely one of them in each column of T^1. Again, using the skew-symmetry of T^1 we may assume that all of the entries in $C_2\cap C_3$ are contained in the first row of T^1. We can further assume by symmetry that they appear in order: $1,2,\cdots,\hat{r}$; this might require changing the order of $\{1,\cdots,\hat{r}\}$ in the first columns of T^2 and T^3, but we can fix that by using the skew-symmetry of T^2 and T^3. What we've shown so far is that, unless $T=0$, T coincides up to sign with T_0 in the first row of the first tableau, and in the 2-nd and 3-rd tableaux.

Now if for some $i \neq j$, $\{i', j'\}$ or $\{i'', j''\}$ are contained in the same column of T^1, then $T = 0$ by the usual argument: interchange i' with j' everywhere, or i'' with j'', to show that $T = -T$. If this isn't the case, then we can do column permutations in such a way that the 2-nd row of T^1 consists of $\{1', \cdots, s'\}$, while the 3-rd consists of $\{1'', \cdots, s''\}$, in some order. We can now use the symmetry to arrange these entries as in T_0^1, which might affect the order in which they appear in T^2 and T^3. Using the skew-symmetry of T^2 and T^3 we can again fix that order. In any case, we have proved that, unless $T = 0$, $T = \pm T_0$, i.e. T_0 spans $\mathrm{Tab}(\lambda, \underline{d}, \underline{n})$.

We show next that if $\mu \vdash (r+s)$ has exactly r parts, then $\pi_\mu(T_0) = 0$. Write $\mu = (\mu_1, \cdots, \mu_r)$. Recall the way π_μ acts: it partitions the alphabet $\mathcal{A}$ into sets $\mathcal{A}_1, \cdots, \mathcal{A}_r$ of sizes $\mu_1, \cdots, \mu_r$ in all possible ways, and then maps the elements of $\mathcal{A}_i$ to i for $i = 1, \cdots, r$ and sums over the corresponding tableaux. Note that if two elements of some $\mathcal{A}_i$ are in the same column of T_0, then after mapping the entries of $\mathcal{A}_i$ to i we get a 3-tableau with repeated entries in some column, i.e. one that is equal to zero. This is why we are only interested in partitions of $\mathcal{A}$ into sets $\mathcal{A}_i$ such that no two entries of some $\mathcal{A}_i$ are in the same column of T_0. Therefore no two of $\{1, \cdots, \hat{r}, 1', \cdots, s'\}$, as well as no two of $\{1, \cdots, \hat{r}, 1'', \cdots, s''\}$ can be in the same subset $\mathcal{A}_i$ of a partition. This implies that the only interesting μ is $(2^s 1^{r-s})$, and correspondingly the only interesting partitions are (up to reordering), $\mathcal{A}_1 = \{1\}, \cdots, \mathcal{A}_{\hat{r}} = \{\hat{r}\}$, $\mathcal{A}_{\hat{r}+1} = \{1', \sigma(1)''\}, \cdots, \mathcal{A}_r = \{s', \sigma(s)''\}$ for σ a permutation of the set $\{1, \cdots, s\}$ (we write this as $\sigma \in S_s$). We get

$$\pi_\mu(T_0) = \sum_{\sigma \in S_s} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r} \\ \hline 1' & 2' & \cdots & s' & & \\ \cline{1-4} \sigma(1)' & \sigma(2)' & \cdots & \sigma(s)' & & \\ \cline{1-4} \end{array} \otimes \begin{array}{|c|c|} \hline 1 & \sigma(1)' \\ \hline 2 & \sigma(2)' \\ \hline \vdots & \vdots \\ \hline s & \sigma(s)' \\ \hline \vdots & \\ \cline{1-1} \hat{r} & \\ \cline{1-1} 1' & \\ \cline{1-1} \vdots & \\ \cline{1-1} s' & \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1' \\ \hline 2 & 2' \\ \hline \vdots & \vdots \\ \hline s & s' \\ \hline \vdots & \\ \cline{1-1} \hat{r} & \\ \cline{1-1} \sigma(1)' & \\ \cline{1-1} \vdots & \\ \cline{1-1} \sigma(s)' & \\ \cline{1-1} \end{array} .$$

For each σ, we can rearrange the entries $\sigma(1)', \cdots, \sigma(s)'$ in the 2-nd and 3-rd tableaux of the RHS of the above expression, so that they appear in increasing order $1', \cdots, s'$. This has the effect of multiplying the corresponding 3-tableau by the square of $\mathrm{sgn}(\sigma)$, i.e. by 1. We can therefore assume that all the tableaux T on the RHS of the above expression have

$$T^2 = T^3 = \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 & 2' \\ \hline \vdots & \vdots \\ \hline s & s' \\ \hline \vdots \\ \cline{1-1} \hat{r} \\ \cline{1-1} 1' \\ \cline{1-1} \vdots \\ \cline{1-1} s' \\ \cline{1-1} \end{array} .$$

We get

$$\pi_\mu(T_0) = \sum_{\sigma \in S_s} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r} \\ \hline 1' & 2' & \cdots & s' \\ \cline{1-4} \sigma(1)' & \sigma(2)' & \cdots & \sigma(s)' \\ \cline{1-4} \end{array} \otimes T^2 \otimes T^3.$$

Since each of the entries $\{1', \cdots, s'\}$ appears the same number of times (twice) in the tableaux on the RHS, we can apply symmetry (property 3. of Definition 5.1.1), and replace i' by $\sigma^{-1}(i)'$ everywhere. This will affect both T^2 and T^3, but we can then use skew-symmetry of these tableaux to rearrange the entries in the original order (again the effect is multiplying the corresponding 3-tableaux by the square of the signature of σ^{-1}, which is 1). We get

$$\pi_\mu(T_0) = \sum_{\sigma \in S_s} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r} \\ \hline 1' & 2' & \cdots & s' \\ \cline{1-4} \sigma(1)' & \sigma(2)' & \cdots & \sigma(s)' \\ \cline{1-4} \end{array} \otimes T^2 \otimes T^3 =$$

$$\sum_{\sigma^{-1} \in S_s} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r} \\ \hline \sigma(1)' & \sigma(2)' & \cdots & \sigma(s)' \\ \cline{1-4} 1' & 2' & \cdots & s' \\ \cline{1-4} \end{array} \otimes T^2 \otimes T^3.$$

Note that we have changed the indexing set from $\sigma \in S_s$ to $\sigma^{-1} \in S_s$ - of course this only amounts to a reordering of the terms. Using skew-symmetry, we can interchange the 2-nd and 3-rd rows of the 1-st tableaux in the terms on the RHS. This amounts to changing the signs of the corresponding tableaux by $(-1)^s = -1$, since s is odd. We get

$$\pi_\mu(T_0) = \sum_{\sigma^{-1}\in S_s} (-1)^s \cdot \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & \cdots & \hat{r} \\ \hline 1' & 2' & \cdots & s' & \multicolumn{2}{c}{} \\ \cline{1-4} \sigma(1)' & \sigma(2)' & \cdots & \sigma(s)' & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array} \otimes T^2 \otimes T^3 = -\pi_\mu(T_0),$$

i.e. $\pi_\mu(T_0) = 0$, as desired.

To finish up, we need to check that $T_0 \neq 0$. As for the classical Strassen's equations (Section 5.3), we show that the monomial m shows up only with coefficient 1 in $T_0 = c_\lambda \cdot m$, with

$$m = \prod_{i\in\mathcal{A}} z_{(\{a_i\},\{b_i\},\{c_i\})},$$

where a_i, b_i, c_i denote the indices of the boxes of T_0^1, T_0^2 and T_0^3 respectively, that contain an entry equal to i. For example $a_1 = b_1 = c_1 = 1$, $a_{1'} = \hat{r}+1$, $b_{1'} = r+1$, $c_{1'} = 2$, etc. As before, it's enough to show that if for some permutation σ of the alphabet $\mathcal{A}$, $\sigma(T_0)$ is obtained from T_0 by a column permutation σ_C, followed by a row permutation σ_R, then $\mathrm{sgn}(\sigma_C) = 1$.

Note first, by looking at T_0^2, that a column permutation followed by a row permutation can't send a box labeled i'' to one labeled j', i.e. $\sigma(j') \neq i''$ for all $i, j \in \{1, \cdots, s\}$. Similarly, by looking at T_0^3 we conclude that $\sigma(i'') \neq j'$ for all $i, j \in \{1, \cdots, s\}$. This means that the column permutation σ_C restricted to the tableau T_0^1 must be a product of column transpositions, each interchanging an element in the first row of T_0 with one in the second or third row of T_0. Write $\sigma_C = \sigma_C^1 \cdot \sigma_C^2 \cdot \sigma_C^3$, where σ_C^i is the restriction of σ_C to the i-th tableau of T_0, and similarly $\sigma_R = \sigma_R^1 \cdot \sigma_R^2 \cdot \sigma_R^3$. We have

$$\sigma_C^1 = \prod_{i\in I} (i, i')_1 \cdot \prod_{j\in J} (j, j'')_1,$$

where I, J are disjoint subsets of $\{1, \cdots, s\}$, and $(i, i')_1$, $(j, j'')_1$ denote the transpositions interchanging the boxes labeled $\{i, i'\}$ and $\{j, j''\}$ respectively (the significance of the index 1 is that the transpositions occur in the first factor). In particular

$$\mathrm{sgn}(\sigma_C^1) = (-1)^{|I|+|J|}.$$

It follows now that, since $\sigma = \sigma_R \cdot \sigma_C$, we must have that σ_R^2 is a product of transpositions interchanging boxes labeled by elements in J with boxes labeled by elements in $J'' = \{j'' : j \in J\}$. Similarly, σ_R^1 is a product of transpositions interchanging boxes labeled by I with those labeled by I' (in some order). We may therefore write

$$\sigma_R^k \cdot \sigma_C^k = \tau_C^k \cdot \tau_R^k,\ k = 2, 3,$$

where

$$\tau_R^2 = \prod_{j\in J}(j,j'')_2,\ \tau_R^3 = \prod_{i\in I}(i,i')_3,$$

$$\tau_C^2 = \hat{\tau}_C^2 \cdot \prod_{i\in I}(i,i')_2,\ \tau_C^3 = \hat{\tau}_C^3 \cdot \prod_{j\in J}(j,j'')_3,$$

for some column permutations $\hat{\tau}_C^2, \hat{\tau}_C^3$.

Note that $\mathrm{sgn}(\sigma_R^k) = \mathrm{sgn}(\tau_R^k)$, $k = 2,3$, since the given permutations are products of the same number of transpositions ($|J|$ and $|I|$ respectively). This means that $\mathrm{sgn}(\sigma_C^k) = \mathrm{sgn}(\tau_C^k)$, and moreover

$$\mathrm{sgn}(\sigma_C^2) = \mathrm{sgn}(\hat{\tau}_C^2) \cdot (-1)^{|I|},$$
$$\mathrm{sgn}(\sigma_C^3) = \mathrm{sgn}(\hat{\tau}_C^3) \cdot (-1)^{|J|}.$$

It follows that

$$\mathrm{sgn}(\sigma_C) = \mathrm{sgn}(\sigma_C^1) \cdot \mathrm{sgn}(\sigma_C^2) \cdot \mathrm{sgn}(\sigma_C^3) = (-1)^{|I|+|J|} \cdot \mathrm{sgn}(\hat{\tau}_C^2) \cdot (-1)^{|I|} \cdot \mathrm{sgn}(\hat{\tau}_C^3) \cdot (-1)^{|J|}$$

$$= \mathrm{sgn}(\hat{\tau}_C^2) \cdot \mathrm{sgn}(\hat{\tau}_C^3).$$

Observe now that

$$\sigma_C^1 \cdot (\prod_{i\in I}(i,i')_2 \cdot \tau_R^2) \cdot (\prod_{j\in J}(j,j'')_3 \cdot \tau_R^3)(T_0) = \tau(T_0),$$

for some permutation τ of the alphabet $\mathcal{A}$, and that the condition

$$\sigma_R \cdot \sigma_C(T_0) = \sigma(T_0),$$

can be rewritten as

$$\sigma_R^1 \cdot \hat{\tau}_C^2 \cdot \hat{\tau}_C^3 \cdot \tau(T_0) = \sigma(T_0).$$

This means that the three permutations σ_R^1, $\hat{\tau}_C^2$ and $\hat{\tau}_C^3$ induce the same permutation of the alphabet $\mathcal{A}$, in particular $\mathrm{sgn}(\hat{\tau}_C^2) = \mathrm{sgn}(\hat{\tau}_C^3)$, and thus their product, which equals $\mathrm{sgn}(\sigma_C)$, is equal to 1. We get that $\mathrm{sgn}(\sigma_C) = 1$, which is the same as saying that the coefficient of m in $c_\lambda \cdot m$ corresponding to the permutation σ is equal to 1, finishing the proof of the proposition. □

5.5 The Salmon Problem: finding the defining ideal of $\sigma_4(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$

The known equations for $\sigma_4(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)) = \sigma_4(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$, where $\dim(V_i) = 4$, $i = 1,2,3$, consist of ([LM08],[BO10])

- the generalized Strassen's equations, of degree 5, with $r = 3$, $s = 1$, i.e. the module

$$S_{(3,1,1)}V_1 \otimes S_{(2,1,1,1)}V_2 \otimes S_{(2,1,1,1)}V_3,$$

  together with its permutations.

- a module of equations of degree 9, isomorphic to

$$S_{(3,3,3)}V_1 \otimes S_{(3,3,3)}V_2 \otimes S_{(3,3,3)}V_3,$$

  which we won't discuss here.

- the module of equations of degree 6

$$S_{(3,1,1,1)}V_1 \otimes S_{(2,2,2)}V_2 \otimes S_{(2,2,2)}V_3,$$

  together with its permutations.

In what follows, we shall give the generic version of the latter. We are in the situation of Example 5.1.3, with $n = 3$, $d_1 = d_2 = d_3 = 1$, $r = 6$, $\lambda^1 = (3, 1, 1, 1)$, $\lambda^2 = \lambda^3 = (2, 2, 2)$. We prove that $\mathrm{Tab}(\lambda, \underline{d}, \underline{n})$ is one dimensional, generated by

$$T_0 = \begin{array}{|c|c|c|}\hline 1&5&6\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array},$$

and that T_0 gives generic equations for the variety of secant 3-planes of a triple Segre product, i.e. $\pi_\mu(T_0) = 0$ for every partition $\mu \vdash 6$ with exactly 4 parts.

To see that T_0 spans, choose any $T \in \mathrm{Tab}(\lambda, \underline{d}, \underline{n})$. We may assume by symmetry that $T^1 = T_0^1$. Also, using the skew-symmetry on columns, and the shuffling relations, we may assume that T^1 and T^2 are standard, i.e. their entries are increasing along both rows and columns. Let us show first that if any three of $\{1, 2, 3, 4\}$ lie in the same column of T^2 or T^3, then $T = 0$. Say this is the case, and the first column of T^2 consists of $\{1, 2, 3\}$. It follows that, since T^3 has only two columns, we can find $\{i, j\} \subset \{1, 2, 3\}$ such that $\{i, j\}$ appear in the same column of each of T^1, T^2 and T^3. The usual argument of switching i with j shows then that $T = 0$.

We assume now that no three of $\{1, 2, 3, 4\}$ appear in the same column of T^2 or T^3. Since we can also assume that T^2, T^3 are standard, and that no pair $\{i, j\}$ appears in the same column of each of T^1, T^2 and T^3, then in fact we see that there are only two possibilities for T, namely

$$T^1 \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array},\quad T^1 \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array}.$$

The first 3-tableau is precisely T_0, while the second one is $-T_0$ (after interchanging $2,3$ everywhere by symmetry, and then interchanging the 2 and 3 in T^1, by skew-symmetry). In any case, we see that T_0 spans. We could give a tedious argument as in the preceding sections to show that $T_0 \neq 0$, but we prefer to use a Macaulay2 calculation ([GS]) using the SchurRings package, which shows that the multiplicity of $S_\lambda V$ inside $S_{(6)}(V_1 \otimes V_2 \otimes V_3)$ is indeed equal to 1.

It remains to prove that $\pi_\mu(T_0) = 0$ whenever $\mu \vdash 6$ has exactly 4 parts. As in Section 5.4, it suffices to concentrate on partitions $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_4$ where no two elements in some $\mathcal{A}_i$ live in the same column of T_0. We see at once that this leaves us with only one choice (up to reordering): $\mu = (2,2,1,1)$ and

$$\mathcal{A}_1 = \{1,6\},\ \mathcal{A}_2 = \{2\},\ \mathcal{A}_3 = \{3\},\ \mathcal{A}_4 = \{4,5\}.$$

We get

$$\pi_\mu(T_0) = \begin{array}{|c|c|c|}\hline 1&4&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 4&1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 4&1\\\hline\end{array}.$$

Let $T' = \pi_\mu(T_0)$. Note that 1 and 4 show up the same number of times in each of the tableaux of T', so we can interchange them by symmetry. We get

$$T' = \begin{array}{|c|c|c|}\hline 4&1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 1\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline 1&4\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline 1&4\\\hline\end{array}.$$

Now interchanging all the pairs $(1,4)$ occurring in the 5 columns of T', we change the sign of the RHS 3-tableau by $(-1)^5 = -1$. We get

$$T' = -\begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 4&1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 4&1\\\hline\end{array}.$$

Using now the shuffling relation applied to the boxes in the 2-nd and 3-rd column of the first tableau of the 3-tableau on the RHS, we get

$$T' = -T',$$

i.e. $T' = 0$, and $\pi_\mu(T_0) = 0$, as desired.

5.6 The Aronhold Invariant

The Aronhold Invariant is a module of equations of degree 4 for the variety of secant planes to the 3-uple embedding of projective space, $\sigma_3(\mathrm{Ver}_3(\mathbb{P}V))$. It is isomorphic as a representation to $S_{(4,4,4)}V$. In what follows, we present the generic version of the Aronhold invariant. We are in the situation $n = 1$, $d_1 = 3$, $r = 4$. We will show that $\mathrm{Tab}(\lambda, \underline{d}, \underline{n})$ is a one-dimensional space, spanned by

$$T_0 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 \\ \hline \end{array},$$

and that $\pi_\mu(T_0) = 0$ for any partition $\mu \vdash 4$ with exactly 3 parts.

We start by proving the last statement: there is only one partition of 4 with 3 parts, namely $\mu = (2, 1, 1)$. Since any pair (i, j) with $i \neq j \in \mathcal{A} = \{1, 2, 3, 4\}$ appears in some column of T_0, it follows that $\pi_\mu(T_0)$ is a sum of tableaux with repeated entries in some column, i.e. $\pi_\mu(T_0) = 0$.

Let now $T \in \mathrm{Tab}(\lambda, \underline{d}, \underline{n})$ be any $(1-)$tableau. Using the shuffling relations and skew-symmetry, we may assume that T is standard, i.e. T has weakly increasing rows and strictly increasing columns. It must then be the case that the first row of T starts as

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}.$$

Since the last column of T must have distinct entries, and T contains only three 1's, it must be that this last column equals

$$\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}.$$

Now there's a unique way to fill in T to a standard tableau, and that yields T_0. To see that T_0 is nonzero, we do again a Macaulay2 calculation ([GS]), showing that the multiplicity of $S_\lambda V$ in $S_{(4)}(S_{(3)}V)$ is equal to 1, hence T_0 must be a basis of $\mathrm{Tab}(\lambda, \underline{d}, \underline{n})$.

Chapter 6

Minors of Catalecticants

In this chapter we prove three cases of Conjecture 1.3.1, namely when $k = 2, 3$ and 4. The case $k = 2$ was already known by work of Pucci, but we give a simpler argument. Except for this case, all the proofs are representation theoretic, and only work in characteristic zero. The basic strategy is to show that for each $k(= 2, 3, 4)$, the equations of degree k of $\sigma_{k-1}(\mathrm{Ver}_d(\mathbb{P}^n))$ coincide with the space of $k \times k$ minors of any of the middle catalecticant matrices (where "middle" here means from the $(k-1)$-st to the $(d-k+1)$-st). It has been proved recently ([BB10]) that for large values of k, $k \times k$ minors of catalecticant matrices aren't enough to generate the ideal of $\sigma_{k-1}(\mathrm{Ver}_d(\mathbb{P}^n))$ even when d is very large. However, we do not know if it will still be the case that for larger values of k ($k \geq 5$), the equations of degree k of $\sigma_{k-1}(\mathrm{Ver}_d(\mathbb{P}^n))$ will coincide with the minors of the middle catalecticants for sufficiently large d.

In Section 6.k, $k = 1, 2, 3$, we shall write $F_{a,b}$ for the space $F_{A,B}^{k+1,k+1}$ introduced in Definition 3.2.11, where $A = (a)$ and $B = (b)$ give a decomposition of $\underline{d} = (d)$ (i.e. $a + b = d$). We refer to $F_{a,b}$ as the space of generic $(k+1) \times (k+1)$-minors of the a-th catalecticant matrix. In general we shall write simply d for the 1-tuple $\underline{d} = (d)$.

6.1 2×2 Minors

In this section we give two proofs of the following result of Pucci, which is the case $k = 2$ of Conjecture 1.3.1. The first proof works in arbitrary characteristic, while the second one is a characteristic zero proof meant to illustrate the methods that we shall use in the case of higher minors.

Theorem 6.1.1 ([Puc98]). *Let K be a field of arbitrary characteristic and let $n, d \geq 2$ be integers. For all t with $1 \leq t \leq d-1$ we have*

$$I_2(Cat(1, d-1; n)) = I_2(Cat(t, d-t; n)).$$

Proof in arbitrary characteristic. For multisets m_1, m_2, n_1, n_2 we let

$$[m_1, m_2|n_1, n_2] = \begin{vmatrix} z_{m_1 \cup n_1} & z_{m_1 \cup n_2} \\ z_{m_2 \cup n_1} & z_{m_2 \cup n_2} \end{vmatrix}.$$

With this notation, we have the following identity for multisets u_1, u_2, v_1, v_2, α_1, α_2, β_1, β_2:

$$\begin{aligned}[u_1 \cup u_2, v_1 \cup v_2|\alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2] = [&u_1 \cup \alpha_1, v_1 \cup \beta_1|u_2 \cup \alpha_2, v_2 \cup \beta_2] \\ &+ [u_1 \cup \beta_2, v_1 \cup \alpha_2|v_2 \cup \alpha_1, u_2 \cup \beta_1].\end{aligned} \tag{6.1.1}$$

We shall prove that $I_2(Cat(a, b; n)) \subset I_2(Cat(a + 1, b - 1; n))$ for $a + b = d$ and $1 \leq a \leq d - 2$. This is enough to prove the equality of the 2×2 minors of all the catalecticants, since $I_2(Cat(1, d - 1; n)) = I_2(Cat(d - 1, 1; n))$. Since the ideal $I_2(Cat(a, b; n))$ is generated by minors $[m_1, m_2|n_1, n_2]$ with $|m_1| = |m_2| = a$ and $|n_1| = |n_2| = b$, it follows from 6.1.1 that it's enough to decompose m_1, m_2, n_1, n_2 as

$$m_1 = u_1 \cup u_2, \quad m_2 = v_1 \cup v_2, \quad n_1 = \alpha_1 \cup \alpha_2, \quad n_2 = \beta_1 \cup \beta_2,$$

in such a way that

$$\begin{aligned} &|u_1| + |\alpha_1| = |v_1| + |\beta_1| = a + 1, \quad |u_2| + |\alpha_2| = |v_2| + |\beta_2| = b - 1, \\ &|u_1| + |\beta_2| = |v_1| + |\alpha_2| = b - 1, \quad |v_2| + |\alpha_1| = |u_2| + |\beta_1| = a + 1, \end{aligned} \tag{6.1.2}$$

or

$$\begin{aligned} &|u_1| + |\alpha_1| = |v_1| + |\beta_1| = a + 1, \quad |u_2| + |\alpha_2| = |v_2| + |\beta_2| = b - 1, \\ &|u_1| + |\beta_2| = |v_1| + |\alpha_2| = a + 1, \quad |v_2| + |\alpha_1| = |u_2| + |\beta_1| = b - 1. \end{aligned} \tag{6.1.3}$$

If $a \leq 2b - 2$, then we can find $0 \leq x, y \leq b - 1$ with $x + y = a$. Choose any such x, y and decompose

$$m_1 = u_1 \cup u_2, \quad m_2 = v_1 \cup v_2, \quad \text{with } |u_2| = |v_1| = x \text{ and } |u_1| = |v_2| = y,$$

and

$$\begin{gathered} n_1 = \alpha_1 \cup \alpha_2, \quad n_2 = \beta_1 \cup \beta_2, \quad \text{with} \\ |\alpha_1| = x + 1,\ |\beta_1| = y + 1,\ |\alpha_2| = b - 1 - x \text{ and } |\beta_2| = b - 1 - y. \end{gathered}$$

It's easy to see then that 6.1.2 is satisfied.

If $b \leq 2a + 2$, then since $b \geq 2$ $(a \leq d - 2)$, we can find $1 \leq x, y \leq a + 1$ with $x + y = b$. Choose any such x, y and decompose

$$n_1 = \alpha_1 \cup \alpha_2, \quad n_2 = \beta_1 \cup \beta_2, \quad \text{with } |\alpha_2| = |\beta_1| = x \text{ and } |\alpha_1| = |\beta_2| = y,$$

and

$$m_1 = u_1 \cup u_2, \quad m_2 = v_1 \cup v_2, \quad \text{with}$$
$$|u_1| = a+1-y,\ |v_1| = a+1-x,\ |u_2| = y-1 \text{ and } |v_2| = x-1.$$

It's easy to see then that 6.1.3 is satisfied.

If neither of $a \leq 2b-2$ and $b \leq 2a+2$ holds, then

$$a \geq 2b-1 \geq 2(2a+3)-1 = 4a+5,$$

so $0 \geq 3a+5$, a contradiction. $\square$

Proof in characteristic zero. By Proposition 3.3.5, it's enough to treat the "generic case". We want to show that for positive integers a, b with $a+b=d$, and $N=2d$, all S_N-subrepresentations $F_{a,b} \subset U_2^d = \mathrm{ind}_{S_d \sim S_2}^{S_N}(\mathbf{1})$ are the same. Clearly the trivial representation $[(N)]$ is not contained in any $F_{a,b}$, so

$$F_{a,b} \subseteq U_2^d/[(N)] = \bigoplus_{i=1}^{\lfloor d/2 \rfloor} [(2\cdot(d-i), 2\cdot i)], \quad \text{for all} \quad a, b \text{ with } a+b=d.$$

(see [Mac95, I.8, Ex. 6] for the formula of the decomposition of U_2^d into irreducible representations; as the rest of the proof will show, we don't really need the precise description of this decomposition).

We will finish the proof by showing that all of the above inclusions are in fact equalities. To see this, it's enough to prove that for any a, b with $a+b=d$, any partition λ with two parts, and any monomial $m = z_\alpha \cdot z_\beta$, with $\alpha \sqcup \beta = \{1, \cdots, N\}$, we have $c_\lambda \cdot m \in F_{a,b}$. Fix then such $a, b, \lambda = (\lambda_1, \lambda_2)$ and $m = z_\alpha \cdot z_\beta$.

Recall from Section 3.2 that we can identify $c_\lambda \cdot m$ with a tableau T of shape λ with 1's in the positions indexed by the elements of α, and 2's in the positions indexed by the elements of β. Recall also that if T has repeated entries in a column, then $T=0$. Since permutations within columns of T can only change the sign of T, and permutations of the columns of T of the same size don't change the value of T (Lemma 3.2.10), we can assume in fact that $m = z_{\{1,\cdots,d\}} \cdot z_{\{d+1,\cdots,N\}}$ and

$$T = c_\lambda \cdot z_{\{1,\cdots,d\}} \cdot z_{\{d+1,\cdots,N\}} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & \cdots & 2 & 2 & \cdots \\ \hline 2 & 2 & \cdots \\ \cline{1-3}\end{array}.$$

Consider the sets

$$\alpha_1 = \{2, \cdots, a+1\},\ \alpha_2 = \{1, \cdots, d\} \setminus \alpha_1,\ \beta_1 \text{ and } \beta_2 = \{d+1, \cdots, N\} \setminus \beta_1,$$

where β_1 is any subset with a elements of $\{d+1, \cdots, N\}$ containing $\lambda_1 + 1$. Let $\tilde{T}$ be the tableau obtained from T by circling the boxes corresponding to the entries of α_1 and β_1 (see Definition 3.2.13). We have

$$\tilde{T} = c_\lambda \cdot [\alpha_1, \beta_1 | \alpha_2, \beta_2] = c_\lambda (z_{\alpha_1 \cup \alpha_2} \cdot z_{\beta_1 \cup \beta_2} - z_{\alpha_1 \cup \beta_2} \cdot z_{\alpha_2 \cup \beta_1}) = T - T',$$

where T' is a tableau with two equal entries in its first column, i.e. $T' = 0$. We get

$$T = \tilde{T} \in F_{a,b},$$

completing the proof. □

Remark 6.1.2. The characteristic zero case also follows by inheritance (Propositions 3.1.6 and 3.3.5): since all the partitions λ that show up have at most two parts, it suffices by inheritance to prove the theorem when $n = 2$, but in this case all the catalecticant ideals are the same, as remarked in the introduction (1.3.1).

6.2 3×3 Minors

We are now ready to give an affirmative answer to Geramita's questions *Q5a* and *Q5b* in the introduction.

Theorem 6.2.1. *Let K be a field of characteristic 0 and let $n \geq 2$, $d \geq 4$ be integers. The following statements hold:*

1. *For all t with $2 \leq t \leq d-2$ we have*

$$I_3(Cat(2, d-2; n)) = I_3(Cat(t, d-t; n)).$$

2. *There is a strict inclusion*

$$I_3(Cat(1, d-1; n)) \subsetneq I_3(Cat(2, d-2; n)).$$

3. *Any of the ideals $I_3(Cat(t, d-t; n))$, $2 \leq t \leq d-2$, is the ideal of the first secant variety to the d-th Veronese embedding of $\mathbb{P}_K^{n-1}$.*

Proof. To prove (1), it suffices by Proposition 3.3.5 to show that $F_{2,d-2} = F_{t,d-t} \subset U_3^d$ for $2 \leq t \leq d-2$. The λ-highest weight spaces of all $F_{t,d-t}$, $2 \leq t \leq d-2$, are the same when λ has at most two parts. This follows by inheritance: combine Proposition 3.3.5 with the fact that the theorem is known when $n = 2$ (1.3.1). We shall prove that when λ has three parts, the λ-part of $F_{t,d-t}$ is equal to the λ-part of U_3^d for all t with $1 \leq t \leq d-1$ (we already know this when $t = 1$, by Proposition 3.2.14). This will imply (1) and the inclusion of (2). The reason why this inclusion is strict for $d \geq 4$ is because it is already strict when $n = 2$, and because inheritance holds for catalecticant ideals.

Consider a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with 3 parts, a monomial $m \in U_3^d$ with corresponding tableau $T = c_\lambda \cdot m$, and integers $2 \leq a \leq b$ with $a + b = d$. We shall prove that $T \in F_{a,b}$. We will see that if λ has only one entry in the second column, then $T = 0$, so let's assume this isn't the case for the moment. We may also harmlessly assume that T has no repeated entries in a column. Since permuting the numbers $1,2,3$ in the tableau T doesn't change T, and permutations within the columns of T preserve T up to sign, we may assume that T contains the subtableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

in its first two columns (there may or may not be a third box in the second column of λ).

It follows that $m = z_{\gamma_1} z_{\gamma_2} z_{\gamma_3}$, with $\gamma_1 = \{1, 2, \cdots\}$, $\gamma_2 = \{\lambda_1 + 1, \lambda_1 + 2, \cdots\}$, $\gamma_3 = \{\lambda_1 + \lambda_2 + 1, \cdots\}$, $|\gamma_i| = d$. Consider subsets $\alpha_i \subset \gamma_i$, $|\alpha_i| = a$ satisfying the conditions

$$1, 2 \in \alpha_1, \quad \lambda_1 + 1 \in \alpha_2, \lambda_1 + 2 \notin \alpha_2, \quad \lambda_1 + \lambda_2 + 1 \notin \alpha_3,$$

and let $\beta_i = \gamma_i \setminus \alpha_i$, for $i = 1, 2, 3$. Let $\tilde{T}$ be the tableau obtained from T by circling the entries of $\alpha_1, \alpha_2, \alpha_3$, so that $\tilde{T} \in F_{a,b}$. $\tilde{T}$ looks like

$$\tilde{T} = \begin{array}{|c|c|c|}\hline ① & ① & \cdots \\ \hline ② & 2 & \cdots \\ \hline 3 & \cdots & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

We get

$$\tilde{T} = c_\lambda \cdot [\alpha_1, \alpha_2, \alpha_3 | \beta_1, \beta_2, \beta_3] = T + \sum_{j=1}^{5} \pm T_j,$$

where each T_j is a tableau with repeated entries in one of its first two columns (i.e. $T_j = 0$). It follows that

$$T = \tilde{T} \in F_{a,b},$$

which is what we wanted to prove.

To see that $T = c_\lambda \cdot m = 0$ for all monomials m when $\lambda = (3d-2, 1, 1)$, it suffices to notice that if σ is the transposition of the $(3d-1)$-st and $3d$-th boxes of λ (the 2nd and 3rd boxes in the first column of λ), then $\sigma(T) = c_\lambda \cdot (\sigma \cdot m)$ and T are the same up to permutations of columns size 1 (and permutations of the entries of the alphabet $\mathcal{A} = \{1, 2, 3\}$). It follows that

$$c_\lambda \cdot m = c_\lambda \cdot (\sigma \cdot m) = (c_\lambda \cdot \sigma) \cdot m = -c_\lambda \cdot m,$$

so that $T = c_\lambda \cdot m = 0$, as desired. Alternatively, see [Mac95, I.8, Ex. 9] for a description of the decomposition of U_3^d into a sum of irreducible representations.

As mentioned in the introduction, part (3) follows from (1), (2) and the result of Kanev ([Kan99], see also [Lan, Corollary 6.4.2.4], or the $(n =)$1-factor case of Theorem 4.1.1). We include a short argument for completeness: by Propositions 3.2.14 and 3.3.5, for λ a partition with at least 3 parts the λ-part of $S_{(r)} S_{(d)} V$ is contained in $I_3(1, d-1; n)$, hence by part (2) also in $I_3(t, d-t; n)$ for all t. It is also contained in the ideal of $\sigma_2(\mathrm{Ver}_d(\mathbb{P}V^*))$, by the last part of 3.1.6. It remains to check that the modules corresponding to partitions with at most 2 parts in the ideal of $\sigma_2(\mathrm{Ver}_d(\mathbb{P}V^*))$ are the same as those in $I_3(t, d-t; n)$ for $2 \le t \le d-2$, but this follows by inheritance from the case of the rational normal curve (1.3.1). $\square$

6.3 4×4 Minors

In this section, we prove Conjecture 1.3.1 in the case $k = 4$.

Theorem 6.3.1. *Let F be a field of characteristic 0 and let $n \geq 2$, $d \geq 6$ be integers. The following statements hold:*

1. *For all t with $3 \leq t \leq d-3$ one has*
$$I_4(Cat(3, d-3; n)) = I_4(Cat(t, d-t; n)).$$

2. *If $n \geq 3$, then there exist strict inclusions*
$$I_4(Cat(1, d-1; n)) \subsetneq I_4(Cat(2, d-2; n)) \subsetneq I_4(Cat(3, d-3; n)).$$

Proof. By the polarization-specialization technique of Section 3.3, it suffices to prove the theorem in the generic case. Namely, we want to show that $F_{3,d-3} = F_{t,d-t}$ for $3 \leq t \leq d-3$ and that $F_{1,d-1} \subsetneq F_{2,d-2} \subsetneq F_{3,d-3}$. Furthermore, it suffices to prove these relations for the λ-parts of the corresponding representations, one partition λ at a time. When λ is a partition with at most two parts, the theorem follows by inheritance from the case $n = 2$ of the rational normal curve (1.3.1). Note that this yields the strict inclusion $F_{2,d-2} \subsetneq F_{3,d-3}$, but does not distinguish between $F_{1,d-1}$ and $F_{2,d-2}$. When $n = 3$, the catalecticant matrix $Cat(1, d-1; n)$ has three rows, so $I_4(Cat(1, d-1; n)) = 0$, whereas $Cat(2, d-2; n)$ has at least 4 rows and 4 columns. It is then easy to see that $I_4(Cat(2, d-2; n))$ is nonzero, so that
$$I_4(Cat(1, d-1; 3)) \subsetneq I_4(Cat(2, d-2; 3)).$$
The strict inclusion for $n > 3$ follows by inheritance.

One important tool that we'll be exploiting throughout the proof of Theorem 6.3.1 is Corollary 3.2.16. We shall also make use of the basic relations of Lemma 3.2.10.

Lemma 6.3.2. *Consider a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$ with $\lambda_3 = 1$, and the tableau $T \in \mathrm{hwt}_\lambda(U_4^d)$,*
$$T = \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots \\ 2 & 2 & \cdots & 2 & 3 & \cdots & 3 & 4 & \cdots & 4 \\ 3 \\ * \end{ytableau},$$
where the entry $*$ appears only if $\lambda_4 = 1$, in which case it is equal to 4. Let's assume that T has a columns equal to $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$, b columns equal to $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$ and c columns equal to $\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$. If $a = b$ then $T = 0$, otherwise T is a linear combination of tableaux T', where each T' is a tableau whose first column and $\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$-columns are the same as those of T, and which also contains the column $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$ and at least $\min(a, b)$ of each of the columns $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$.

Proof. Suppose first that $a = b$. Interchanging the 2's and the 3's in T yields a tableau $T' = T \in \mathrm{hwt}_\lambda(U_4^d)$, but since T' differs from T by the transposition of the 2-nd and 3-rd entries in the first column, and by permutations of columns of the same size, it follows from parts (1) and (2) of Lemma 3.2.10 that

$$T = T' = -T, \text{ i.e. } T = 0.$$

Assume now that $a > b$. Since T has the same number (d) of entries equal to 2 and 3, there must be a column of size one of T whose entry equals 3. It follows that T contains the subtableau

$$S = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau},$$

and thus we can use the shuffling relation (3) of Lemma 3.2.10,

$$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} + \begin{ytableau} 3 & 1 \\ 2 \end{ytableau} = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} - \begin{ytableau} 2 & 1 \\ 3 \end{ytableau},$$

to write

$$T = T_1 - T',$$

where T_1 is obtained from T by interchanging the 2 and 3 in S, and T' is obtained from T by applying the permutation $(1, 2, 3)$ (in cycle notation) to the entries in S. It follows that

$$T \equiv T_1 \bmod U,$$

where U is the subspace of $\mathrm{hwt}_\lambda(U_4^d)$ spanned by tableaux T', with T' containing the column $\begin{ytableau} 2 \\ 3 \end{ytableau}$ and at least $\min(a, b)$ of each of $\begin{ytableau} 1 \\ 2 \end{ytableau}$ and $\begin{ytableau} 1 \\ 3 \end{ytableau}$, as in the statement of the lemma. Repeating this process, we obtain after s steps that

$$T \equiv T_s \pmod U,$$

where T_s is obtained from T by interchanging $(a - b)$ 2's from the $\begin{ytableau} 1 \\ 2 \end{ytableau}$-columns with $(a - b)$ 3's from the columns of T of size one. If we now interchange the 2's and 3's in T_s, we get a tableau that differs from T by a transposition in the first column and permutations of columns of the same size. It follows that

$$T_s = -T,$$

so that

$$T \equiv -T \pmod U \iff 2T \in U \iff T \in U.$$

$\square$

6.3.1 Diagrams with 4 rows

Lemma 6.3.3. *For every partition λ of $4d$ with four parts, the λ-part of U_4^d is contained in, and hence equal to, the λ-part of each of $F_{i,d-i}$, for $i = 1, \cdots, d-1$.*

Proof. We know this already for $i = 1$ (and hence also for $i = d - 1$) by Proposition 3.2.14. We fix i with $2 \leq i \leq d/2$ for the rest of the proof. We want to show that any tableau $T \in \mathrm{hwt}_\lambda(U_4^d)$ of shape λ is contained in $F_{i,d-i}$. We have several cases, according to the shape of λ.

Our basic strategy will be to show that $T = \tilde{T}$, where $\tilde{T}$ is obtained from T by circling i entries of each of $1, 2, 3, 4$. We shall always picture a subset of the entries of T, some of which will be circled (at most i of each of $1, 2, 3, 4$). The understanding will be that there are other circled entries, to make up for a total of i circled 1's, 2's, 3's and 4's. It will always be the case that the equality $T = \tilde{T}$ will be a consequence of the fact that any permutation $\sigma \in S_4$ applied to the circled entries will produce a repetition in one of the shown columns.

(1) The first two columns of λ have size 4.

Without loss of generality, we may assume the first 2 columns of T are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array}.$$

The reason for this is because if T had repetitions in one of its columns, then it would be zero. Otherwise, the entries in each of its first two columns have to be $\{1, 2, 3, 4\}$, in some order. We can rearrange them (at the cost of maybe changing the sign of T, according to relation (1) of Lemma 3.2.10) so that we get the columns pictured above. In general, we will argue without loss of generality for the choice of the pictured entries of T, and the reasoning behind it will be along the lines of the argument just presented. This is why we shall skip most of the details in the future cases. We let

$$\tilde{T} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array}.$$

Let us check that $T = \tilde{T}$ in this case (again, the following cases will be very similar and we ask the reader to fill in the necessary details). We have

$$\tilde{T} = \sum_{\sigma \in S_4} \mathrm{sgn}(\sigma) \cdot \sigma(T).$$

We claim that if σ is not the identity, then $\sigma(T)$ has repeated entries in one of its first two columns, and is thus equal to zero: if $\sigma(1) = 3$ or $\sigma(1) = 4$, we get a repetition in the first column, while if $\sigma(1) = 2$ we get a repetition in the second column. We may thus assume that $\sigma(1) = 1$. If $\sigma(2) = 3$ or $\sigma(2) = 4$, then we get a repetition in the first column, so we may assume that $\sigma(2) = 2$. If $\sigma(3) = 4$, then we get a repetition in the second column, thus we assume that $\sigma(3) = 3$ which forces σ to be the identity.

(2) We can find two columns of λ, different from the first one, which contain two pairs of numbers forming a partition of $\{1, 2, 3, 4\}$.

We may assume that the two pairs are $\{1, 2\}$ and $\{3, 4\}$, in which case we let

$$\tilde{T} = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & & \\ 4 & & \end{array}.$$

(3) The second column of λ has size 3, the third column of λ has size at least 2, and (2) doesn't happen.

Up to renumbering the entries of T and column permutations, we have one of

$$\tilde{T} = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & \\ 4 & & \end{array} \quad \text{or} \quad \tilde{T} = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 3 & 3 & \\ 4 & & \end{array}.$$

(4) The second column of λ has size 3, and all the others have size 1.

We may assume that the first three columns of T are

$$\begin{array}{ccc} 1 & 1 & 4 \\ 2 & 2 & \\ 3 & 3 & \\ 4 & & \end{array}.$$

By part (3) of Lemma 3.2.10 applied with $t = 4$, C the second column of T, and $b = 4$, the entry of the unique box in the third column of T, we get

$$T = \begin{array}{ccc} 1 & 4 & 1 \\ 2 & 2 & \\ 3 & 3 & \\ 4 & & \end{array} + \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 4 & \\ 3 & 3 & \\ 4 & & \end{array} + \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & 4 & \\ 4 & & \end{array}$$
$$= -\begin{array}{ccc} 4 & 4 & 1 \\ 2 & 2 & \\ 3 & 3 & \\ 1 & & \end{array} - \begin{array}{ccc} 1 & 1 & 2 \\ 4 & 4 & \\ 3 & 3 & \\ 2 & & \end{array} - \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 4 & 4 & \\ 3 & & \end{array} = -3T,$$

so that $T = 0$. The reason why each of the tableaux on the last row are equal to T is because we can permute in each case the entries $\{1, 2, 3, 4\}$ so that we get tableaux that coincide with

T (in their first three columns). But this is enough for those tableaux to be equal to T, by part (2) of Lemma 3.2.10: all the columns that are not shown have size one, so their order is irrelevant.

(5) The second column of λ has size 2.

If T contains the subtableau

$$S = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},$$

then we take

$$\tilde{T} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}$$

We show that the tableaux containing S, together with the ones satisfying (2), span the λ-highest weight space of U_4^d.

If T doesn't satisfy (2), we may assume that it contains the subtableau

$$S' = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}.$$

To see this, note first that the third column of T must have two entries, otherwise $T = 0$: if there is just one entry in the third column of T, we may assume that the first two columns are

$$S'' = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}.$$

Interchanging the 3's and 4's in T preserves the element $T \in \mathrm{hwt}_\lambda(U_4^d)$, and yields a tableau differing from T by a transposition in its first column, and permutations of columns (of size one). This shows that $T = -T$, i.e. $T = 0$. We may thus assume that the third column of T has size two. We may still assume that the first two columns of T are given by S''. Using the relation

$$\begin{array}{|c|c|}\hline a&1\\\hline \multicolumn{1}{|c|}{b}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&a\\\hline \multicolumn{1}{|c|}{b}\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline a&b\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&a\\\hline \multicolumn{1}{|c|}{b}\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&b\\\hline \multicolumn{1}{|c|}{a}\\\cline{1-1}\end{array},$$

we may assume that the third column of T has the form $\begin{array}{|c|}\hline 1\\\hline *\\\hline\end{array}$. If $* = 4$, then we interchange the 3's and 4's in T and get that T contains

$$\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 \\ 3 \end{ytableau}$$

which is S' up to sign (transposing 3 and 4 in the first column). Now if $* = 1$ then $T = 0$, while if $* = 3$ then T contains S'. If $* = 2$, then using the relation

$$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = \begin{ytableau} 3 & 1 \\ 2 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = -\begin{ytableau} 2 & 1 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 \end{ytableau},$$

we may assume that the third column of T is either $\begin{ytableau} 1 \\ 3 \end{ytableau}$ (so that T contains S'), or $\begin{ytableau} 2 \\ 3 \end{ytableau}$. Interchanging the 1's and 2's in T in the latter case, it follows that T contains

$$\begin{ytableau} 2 & 2 & 1 \\ 1 & 1 & 3 \\ 3 \\ 4 \end{ytableau} = (-1)^2 \cdot \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 \\ 4 \end{ytableau} = S'.$$

We may thus assume that T contains S'. If all the other columns of T have size 1, then $T = 0$ by Lemma 6.3.2. If T contains the column $\begin{ytableau} 2 \\ 3 \end{ytableau}$ then T contains S, which is the case already discussed. If T contains one of $\begin{ytableau} 2 \\ 4 \end{ytableau}$ or $\begin{ytableau} 3 \\ 4 \end{ytableau}$, then we're in case (2). Otherwise, every column of T is one of $\begin{ytableau} 1 \\ 2 \end{ytableau}$, $\begin{ytableau} 1 \\ 3 \end{ytableau}$ or $\begin{ytableau} 1 \\ 4 \end{ytableau}$, so Lemma 6.3.2 applies to give the desired conclusion. □

6.3.2 Diagrams with 3 rows

Throughout the rest of this chapter μ denotes the partition $(2, 1, 1)$ of 4. U_μ^d is the S_{4d}-representation introduced in Definition 3.2.4. For a partition $\lambda \vdash 4d$, $\mathrm{hwt}_\lambda(U_\mu^d)$ is the space of tableaux of shape λ containing d 1's, d 2's and $2d$ 3's (see Definition 3.2.8). Recall that a standard tableaux is one that has weakly increasing rows and strictly increasing columns. We have the following

Proposition 6.3.4. *Let λ be a partition of $4d$ with (at most) three parts. The vector space $\mathrm{hwt}_\lambda(U_\mu^d)$ has a basis consisting of the standard tableaux with an even number of columns containing* $\begin{ytableau} 1 \\ 2 \end{ytableau}$.

Proof. Recall that U^d_μ is the generic version of the more familiar $GL(V)$-representation $S_{(2)}(S_{(d)}V) \otimes S_{(2d)}V$, so the two spaces decompose in the same way into irreducible representations (with respect to the combinatorial data given by partitions of $4d$). We have ([FH91] or [Mac95])

$$S_{(2)}(S_{(d)}V) = \bigoplus_{\substack{\rho \vdash 2d \\ \rho_2 \text{ even}}} S_\rho V.$$

It follows for the generic version $U^d_2 (= U^d_{(1,1)})$ of $S_{(2)}(S_{(d)}V)$ that

$$U^d_2 = \bigoplus_{i \leq d/2} [(2d-2i, 2i)],$$

and that a basis for $\mathrm{hwt}_{(2d-2i,2i)}(U^d_2)$ consist of the tableaux

$$T(i) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \cdots & 1 & 1 & \cdots & 2 & \cdots \\ \hline 2 & \cdots & 2 \\ \cline{1-3} \end{array}$$

with $2i$ columns equal to $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$. $S_{(2d)}V$ is irreducible, and so is its generic version, the trivial representation $U^d_{(2)}$ of S_{2d}, having a basis consisting of the tableau

$$t = \begin{array}{|c|c|c|c|} \hline 3 & 3 & \cdots & 3 \\ \hline \end{array}$$

(in order to stick to our usual notation, we should be writing 1's instead of 3's for the entries of the above tableau, but the 3's are more suggestive for what is to follow). Pieri's formula now implies that

$$S_{(2)}(S_{(d)}V) \otimes S_{(2d)}V = \bigoplus (S_\lambda V)^{m_\lambda},$$

where m_λ is the number of ways of obtaining a tableau of shape λ by starting with one of the $T(i)$'s, and adding $2d$ 3's, no two in the same column. This procedure yields precisely all the standard tableaux of shape λ, with d 1's, d 2's and $2d$ 3's, having an even number of columns containing $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$. It follows that these standard tableaux form a basis for $\mathrm{hwt}_\lambda(U^d_4)$, as long as we can prove that they span.

Since our tableaux satisfy skew-symmetry on rows, and the usual shuffling relations, it follows that standard tableaux span $\mathrm{hwt}_\lambda(U^d_4)$. Consider now a standard tableau T, with an odd number of columns containing $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$, say a. We prove that T is a linear combination of tableaux with at least $a+1$ columns containing $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$. The argument is similar to that of Lemma 6.3.2.

Suppose that T contains b columns equal to $\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}$ and c columns equal to $\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$. We have $b \geq c$, since there's the same number of 1's and 2's in T, and there can be no 1 to the right of a column equal to $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$, because T is standard. Also, there are at least $b-c$ columns of T of size one with their entry equal to 2. Using the relation

$$S = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}$$

on a subtableau S of T consisting of a $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$-column of T and a $\begin{array}{|c|}\hline 2\\\hline\end{array}$-column of T, we can write

$$T = T' + T_1,$$

where T' has $(a+1)$ $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$-columns, and T_1 has $(b-1)$ $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$-columns and $(c+1)$ $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$-columns. Repeating this procedure $(b-c)$ times, we get that $T \equiv T_{b-c}$ modulo the space U_{a+1} of tableaux with at least $(a+1)$ $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$-columns, and T_{b-c} differs from T by replacing $(b-c)$ of its $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$-columns with $(b-c)$ $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$-columns, and $(b-c)$ of its $\begin{array}{|c|}\hline 2\\\hline\end{array}$-columns with $\begin{array}{|c|}\hline 1\\\hline\end{array}$-columns. Interchanging the 1's and 2's in T_{b-c} (and possibly permuting some columns), we recover the tableau T, with the 1's and 2's in its first a columns interchanged. Since a is odd, the skew-symmetry of tableaux implies that $T_{b-c} = -T$, hence

$$T \equiv -T \pmod{U_{a+1}} \Longleftrightarrow T \in U_{a+1},$$

as desired. $\square$

Diagrams with $\lambda_3 \geq 2$

Lemma 6.3.5. *If $T \in U_4^d$ is a tableau containing*

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&4\\\hline 3&4&*\\\hline\end{array}$$

as a subtableau (where $*$ indicates that there might or might not be a box in the corresponding position), then $T \in F_{i,d-i}$ for all $2 \leq i \leq d-2$.

Proof. If there's a box in the $*$-position ($\lambda_3 \geq 3$), we may assume its entry is a 1, in which case T is equal to

$$\tilde{T} = \begin{array}{|c|c|c|}\hline \textcircled{1}&\textcircled{1}&\textcircled{3}\\\hline 2&\textcircled{2}&4\\\hline \textcircled{3}&4&1\\\hline\end{array}.$$

Assume now that there is no box in the $*$-position, and that all the columns of λ, except the first three, have size 1 ($\lambda_2 = 3$). Then interchanging 3 and 4 doesn't change the element

$T \in U_4^d$, but transforms T into a tableau that differs from it by permutations of columns, and a column transposition (the 3 and 4 in the third column). This implies that $T = -T$, i.e. $T = 0$.

We're left with the case when $\lambda_2 > 3$ and $\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$ is a column of T. Applying the usual shuffling relations (3) of Lemma 3.2.10, with C a column of T of size at least two, not in S', and $b = 1$, we may assume that T has a column containing $\begin{array}{|c|}\hline 1\\ \hline a\\ \hline\end{array}$, where $a = 2$ or $a = 3$ (which is equivalent to $a = 4$, up to interchanging the 3's and 4's in T). If $a = 2$, we take

$$\tilde{T} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & \textcircled{1} & \textcircled{3}\\ \hline 2 & 2 & 2 & 4\\ \hline \textcircled{3} & \textcircled{4} & * & \multicolumn{1}{c}{}\\ \cline{1-3}\end{array}.$$

If $a = 3$, we take

$$\tilde{T} = \begin{array}{|c|c|c|c|}\hline 1 & \textcircled{1} & \textcircled{1} & \textcircled{3}\\ \hline \textcircled{2} & \textcircled{2} & 3 & 4\\ \hline 3 & 4 & * & \multicolumn{1}{c}{}\\ \cline{1-3}\end{array}.$$

□

Lemma 6.3.6. *If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $4d$ with three parts and $\lambda_3 \geq 2$, then the restriction of the map $\pi_{(2,1,1)}$ to $\mathrm{hwt}_\lambda(U_4^d)$ surjects onto $\mathrm{hwt}_\lambda(U_\mu^d)$ with kernel $\mathrm{hwt}_\lambda(F_{i,d-i})$ for any $i = 2, \cdots, d-2$. Therefore, the λ-parts of all $F_{i,d-i}$ ($i \neq 1, d-1$) coincide.*

Proof. We first show that $\mathrm{hwt}_\lambda(U_4^d)$ is spanned by tableaux T whose first two columns are

$$S = \begin{array}{|c|c|}\hline 1 & 1\\ \hline 2 & 2\\ \hline 3 & 4\\ \hline\end{array}.$$

These tableaux together with the once whose first two columns are

$$\begin{array}{|c|c|}\hline 1 & 1\\ \hline 2 & 2\\ \hline 3 & 3\\ \hline\end{array}$$

span $\mathrm{hwt}_\lambda(U_4^d)$, so it suffices to show that the latter are linear combinations of the former. We use the shuffling relation (3) of Lemma 3.2.10 with C being the second column of a tableau T of the latter type, and with $b = 4$. We get

$$T = \begin{array}{|c|c|c|}\hline 1&1&4\\ \hline 2&2&\multicolumn{1}{c}{}\\ \cline{1-2} 3&3&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 1&4&1\\ \hline 2&2&\multicolumn{1}{c}{}\\ \cline{1-2} 3&3&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&4&\multicolumn{1}{c}{}\\ \cline{1-2} 3&3&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2&\multicolumn{1}{c}{}\\ \cline{1-2} 3&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 3&4&3\\ \hline 2&2&\multicolumn{1}{c}{}\\ \cline{1-2} 1&1&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 3&4&\multicolumn{1}{c}{}\\ \cline{1-2} 2&2&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2&\multicolumn{1}{c}{}\\ \cline{1-2} 3&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array}$$

$$= \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 3&4\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 3&4\\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 3&4\\ \cline{1-2}\end{array}, \tag{*}$$

where in the above relations we only showed the relevant boxes, and we used freely the symmetry and skew-symmetry of tableaux (as in Definition 5.1.1). Note that the three tableaux in (*) are not the same (in general): they differ in the boxes that are not shown. In any case, we wrote T as a sum of three tableaux containing the subtableau S in their first two columns.

We turn our attention to tableaux of shape λ in $\mathrm{hwt}_\lambda(U^d_\mu)$. We let T_j denote the (unique) standard tableau of shape λ with $2j$ columns containing $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$, for $\lambda_3 \leq 2j \leq \min(\lambda_2, d)$. Proposition 6.3.4 states that the T_j's give a basis for $\mathrm{hwt}_\lambda(U^d_\mu)$. For each T_j, we consider a "lift" L_j, which is any tableau obtained from T_j by replacing the 3 in the 3-rd row and 2-nd column of T_j, together with $(d-1)$ other 3's (not in the first column of T_j), with 4's. We have that $\pi_\mu(L_j) = T_j$ for all j, showing that the lifts L_j constructed in this way are linearly independent modulo $\ker(\pi_\mu)$. Since every $F_{i,d-i}$ is contained in $\ker(\pi_\mu)$, if we can show that the L_j's generate $\mathrm{hwt}_\lambda(U^d_4)$ modulo $F_{i,d-i}$, for $i = 2, \cdots, d-2$, then it follows that in fact we have the equality $\mathrm{hwt}_\lambda(\ker(\pi_\mu)) = \mathrm{hwt}_\lambda(F_{i,d-i})$ for each such i.

We need one more lemma, which will show that the choice of the lifts L_j can be done in an arbitrary way. More generally, once we fix the subtableau consisting of the first two columns of a tableau T to be S, permuting the 3's and 4's in the other columns doesn't change T modulo any of the spaces $F_{i,d-i}$ of generic flattenings.

Lemma 6.3.7. *Let $T \in \mathrm{hwt}_\lambda(U^d_4)$ be any tableau containing S as a subtableau, and let σ be any permutation that interchanges some of the 3's and 4's of T, not contained in S. For any $i = 2, \cdots, d-2$, we have that $T - \sigma(T) \in F_{i,d-i}$.*

Proof. If T has a column containing $\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$, then $T \in F_{i,d-i}$ for all $i = 2, \cdots, d-2$, by Lemma 6.3.5. The same lemma, or the fact that $\sigma(T)$ might have repeated entries in some column, implies that $\sigma(T) \in F_{i,d-i}$, thus $T - \sigma(T) \in F_{i,d-i}$.

Assume now that T doesn't have a column containing $\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$. It's enough to prove the lemma for σ a transposition of two entries, 3 and 4. Assume that the column of 3 is to the left of the column of 4. We apply the shuffling relation (3) of Lemma 3.2.10, with C the column of 3 and $b = 4$. We get

$$T = \sigma(T) + \sum T',$$

where the sum is empty if C has only one entry, and otherwise each T' is a tableau having a column containing $\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$. As we noted above, $T' \in F_{i,d-i}$, hence $T - \sigma(T) \in F_{i,d-i}$. $\square$

To finish the proof of Lemma 6.3.6, we note that because of the preceding lemma, the last part of the proof of Proposition 6.3.4 carries over to show that the lifts L_j generate $\mathrm{hwt}_\lambda(U_4^d/F_{i,d-i})$, for any $i = 2, \cdots, d-2$. Since the L_j's are linearly independent modulo $\ker(\pi_\mu)$, the conclusion follows. $\square$

Diagrams with $\lambda_3 = 1$

In this section, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $4d$ with three parts and $\lambda_3 = 1$. The proof strategy is similar to the one in the previous case. We show that $\mathrm{hwt}_\lambda(U_4^d)$ is spanned by tableaux T containing the subtableau

$$S = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 4 & 4 \\ \hline 3 \\ \cline{1-1}\end{array},$$

and that modulo any $F_{i,d-i}$, any such T is invariant under permutations of 3's and 4's not contained in S. We then show that tableaux T as above, with an even number of columns containing $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, span $\mathrm{hwt}_\lambda(U_4^d/F_{i,d-i})$, and that they map via π_μ to a subset of a basis of U_μ^d. This suffices to conclude that all $\mathrm{hwt}_\lambda(F_{i,d-i})$ coincide with $\mathrm{hwt}_\lambda(\ker(\pi_\mu))$, hence they are the same. We start with the analogue of Lemma 6.3.5:

Lemma 6.3.8. *If $T \in \mathrm{hwt}_\lambda(U_4^d)$ is a tableau containing the subtableau*

$$S' = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 4 & 4 \\ \hline 3 \\ \cline{1-1}\end{array},$$

then $T \in F_{i,d-i}$ for all $i = 2, \cdots, d-2$.

Proof. Suppose first that T has a column of size two not containing 4, say $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. Then we take

$$\tilde{T} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 1 \\ \hline 2 & 4 & 4 & 4 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Otherwise, all the columns of T of size two must contain the entry 4, say a of them also contain 1, b contain 2 and c contain 3. We may assume that $a \geq b$, and apply the (by now standard) argument of Lemma 6.3.2. We use the relation

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$(a-b)$ times, with 2 being the entry of a column of size one. This is going to replace T (modulo the space of tableaux previously analyzed, which is contained in $F_{i,d-i}$) with a tableau T' having a columns equal to $\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array}$ and b columns equal to $\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array}$. Interchanging 1's and 2's in T' we recover T, up to reordering columns and transposing the 1 and 2 entries in its first column. This shows that $T' = -T$, which combined with $T - T' \in F_{i,d-i}$ yields $T \in F_{i,d-i}$. $\square$

Note that Lemma 6.3.7 applies, with the new S defined above: the proof is the same, except for Lemma 6.3.5 being replaced with Lemma 6.3.8.

Lemma 6.3.9. *The tableaux T of shape λ containing S span* $\mathrm{hwt}_\lambda(U_4^d)$.

Proof. We may assume that the first column of a tableau $T \in \mathrm{hwt}_\lambda(U_4^d)$ has entries $\{1, 2, 3\}$, and using the usual shuffling relations, that 4 is contained in its second column. It follows that up to changing the sign of T, its first two columns coincide with those of S. Assume first that T has a column of size 2 not containing 1. Possibly using the shuffling relation

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 4 \\ \cline{1-1}\end{array},$$

we may assume that T also contains one of $\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array}$ or $\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}$. If it contains the former, then T contains S, otherwise we switch the 2's and 3's in T.

Consider now the situation when all the columns of T of size two contain 1. Lemma 6.3.2 implies that T is zero, or a linear combination of tableaux containing the columns $\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$, which as we've seen are in turn combinations of tableaux containing S. $\square$

Lemma 6.3.10. *If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $4d$ with three parts and $\lambda_3 = 1$, then the restriction of the map $\pi_{(2,1,1)}$ to $\mathrm{hwt}_\lambda(U_4^d)$ has kernel $\mathrm{hwt}_\lambda(F_{i,d-i})$ for any $i = 2, \cdots, d-2$. Therefore, the λ-parts of all $F_{i,d-i}$ coincide.*

Proof. We can use the argument of the proof of Proposition 6.3.4 to show that tableaux T containing S and having an even number of columns equal to $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ span $\mathrm{hwt}_\lambda(U_4^d/F_{i,d-i})$. More precisely, for $1 \le 2j \le \min(d, \lambda_2) - 2$, we choose L_j to be a tableau containing S in its first three columns, having a total of $2j$ columns containing $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, and such that the tableau obtained from L_j by removing S is standard. The L_j's form a spanning set for $\mathrm{hwt}_\lambda(U_4^d/F_{i,d-i})$.

We now show that $\pi_\mu(L_j)$ are linearly independent in U_μ^d, which implies that $\mathrm{hwt}_\lambda(F_{i,d-i}) = \mathrm{hwt}_\lambda(\ker(\pi_\mu))$. For $1 \le 2j \le \min(d, \lambda_2)$, we let S_j denote the unique standard tableau in $\mathrm{hwt}_\lambda(U_\mu^d)$ with $2j$ columns containing $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. The S_j's form a basis for $\mathrm{hwt}_\lambda(U_\mu^d)$, by Proposition

6.3.4. We have

$$\pi_\mu(L_j) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & \cdots \\ \hline 2 & 3 & 3 & \cdots \\ \hline 3 \\ \cline{1-1}\end{array}.$$

If $\pi_\mu(L_j)$ doesn't contain a column of size one whose entry is equal to 1, then after reordering its columns of size two, so that the $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$'s come before the $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$'s, which in turn come before the $\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$'s, we obtain the tableau S_j. This shows that $\pi_\mu(L_j) = S_j$.

If $\pi_\mu(L_j)$ contains a column of size one whose entry is 1, we let $b = 1$ be the entry in that column, and C be the third column of $\pi_\mu(L_j)$. Applying the shuffling relation (3) of Lemma 3.2.10, we get

$$\pi_\mu(L_j) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & \cdots \\ \hline 2 & 3 & 3 & \cdots \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & \cdots \\ \hline 2 & 3 & 2 & \cdots \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Up to reordering columns of size two, the first tableau on the RHS of the above relation is equal to S_j, while the second one is a tableau with $2j+1$ columns containing $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$, i.e. by the proof of Proposition 6.3.4, it is a linear combination of $S_{j+1}, S_{j+2}, \cdots$.

Writing the elements $\pi_\mu(L_j)$ as row vectors of coordinates with respect to the basis of S_j's we obtain a matrix in row-echelon form, with a pivot equal to 1 in every row. This matrix has thus maximal rank, showing that the $\pi_\mu(L_j)$'s are linearly independent. $\square$

$\square$

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