Singular Foci of Planar Linkages

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Abstract

The focal points of a curve traced by a planar linkage capture essential information about the curve. Knowledge of the singular foci can be helpful in the design of path-generating linkages and is essential to the determination of path cognates. This paper shows how to determine the singular foci of planar linkages built with rotational links. The method makes use of a general formulation of the tracing curve based on the Dixon determinant of loop equations written in isotropic coordinates. In simple cases, the singular foci can be read off directly from the diagonal of the Dixon matrix, while the worst case requires only the solution of an eigenvalue problem. The method is demonstrated for one inversion each of the Stephenson-3 six-bar and the Watt-1 six-bar.

Key words: Focal points, singular focus, Foci: singular, Linkages: planar, isotropic coordinates, Dixon determinant

1 Introduction

Singular foci play a central role in the kinematics of four-bar coupler curves. For example, in 1875, Roberts [7] proved that every four-bar curve is triply generated, that is, traced by three distinct four-bar linkages. This proof was nonconstructive, but Cayley [2] soon followed with a concrete construction that, given one four-bar, derives the other two cognates. These are now known as *Roberts cognates* [1, pp. 339-341] [4]. Roberts' Theorem hinged on the determination that a four-bar coupler curve has three singular foci. These three foci, two of which are simply the fixed pivots of the linkage, define the "focal triangle," a triangle that is geometrically similar to the coupler triangle.

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Despite the passing of over a century of active kinematic research, it seems that the foci of multiloop planar linkages have never been determined. This paper presents a method of finding the singular foci of any curve traced out by a planar linkage built with rotational joints. The approach uses the expression for the tracing curve from [11], based on the Dixon determinant and isotropic coordinates . This leads to an algorithm for numerically determining the singular foci, which either appear directly on the diagonal of the Dixon matrix, or in worst case, are given by eigenvalues of a matrix extracted from the Dixon matrix. In the case that a singular focus appears with multiplicity greater than one, a method is given for determining its correct multiplicity.

One reason that the singular foci of multiloop linkages have not been studied is that the polynomial describing the tracing curve is quite complicated. Although derivations had been found for each of the six-bar linkages [9], until recently, there did not exist a method for deriving these polynomials for general linkages. This limitation is overcome by the general formulations in [6,10,11] for solving input/output problems: each of these can be adapted to produce tracing curve polynomials. The last of these, [11], gives the tracing curve equation in a particularly convenient form for studying the singular foci. The elegance of this approach is marred by the inclusion in some cases of an extraneous factor. A by-product of the current study is the determination of this factor, so that it can be cancelled out.

As in Roberts' work, one of the main applications of focal points is the determination of *path cognates*, linkages that generate the same curve. In [8], Roth gave geometric constructions for deriving path cognates of certain linkages. These constructions demonstrate existence of path cognates, but do not by themselves prove that all cognates have been found. For completeness results, Roth turned to an algebraic analysis of the coefficients of the tracing curve and obtained completeness results for several geared five-bar linkages. In this paper, we will show that matching the singular foci is equivalent to matching the coefficients of the terms of highest bidegree, as defined herein.

Constructions for path cognates of all of the six-bar linkages are known [4]. (Cognates for body guidance and for function generation are also given in [4], but we do not discuss these in this article.) As in Roth's work, these constructions, which make no overt use of the singular foci, show the existence of path cognates but do not establish completeness. The new understanding of singular foci brought out in this paper will be key to the complete determination of path cognates in future work.

The paper proceeds as follows. First, in §2 we review some basic facts about the geometry of the plane, and in particular the isotropic points of the plane. Then, §3 presents the definition of singular foci of a curve and shows that these are determined by the terms of highest bidegree in the curve's equation. This observation leads in §4 to a determination of the singular foci from the Dixon determinant form of tracing curve equations. Section 5 briefly discusses how the methodology can be extended to treat mechanisms that include prismatic joints. A final section, §6, gives numerical results for a Stephenson-3 six-bar and a Watt-1 six-bar.

2 Background

In this section, some basic facts concerning isotropic points and one- and two-homogeneous treatments of the plane are reviewed. Related expository material can be found in [12]. An algebraic concept, called a support polynomial, is also introduced. All of these concepts are useful in working with singular foci, as we will see in the §3.

2.1 One-homogenization of the plane

The most common method of accounting for how plane curves "meet infinity" is to consider a one-homogenization of the plane. When classical texts in geometry or kinematics speak of "the line at infinity" and related concepts, they often assume this model of infinity without stating so. We first review this concept for Cartesian coordinates and isotropic coordinates, and then look at the isotropic points at infinity.

2.1.1 Cartesian coordinates

Let the Cartesian coordinates of the plane be (x, y). Many results in geometry and kinematics become simpler if one extends the plane to include a line at infinity, having one point for each direction in the plane. The new space is represented by homogeneous coordinates $[X, Y, W]$, where the brackets signify that only the ratios of the coordinates matter. (We do not allow all three coordinates to be zero simultaneously.) For finite points, $W \neq 0$, the correspondence between the two is $(x, y) = (X/W, Y/W)$. In addition to the finite points, the new space has *points at infinity*, defined by $W = 0$. Since this is a linear equation, these points collectively form the *line at infinity*. Considering all of the coordinates to take on complex number values, mathematicians call $(x, y) \in \mathbb{C}^2$ the two-dimensional complex Cartesian space and call $[X, Y, W] \in \mathbb{P}^2$ the two-dimensional complex projective space.

The correspondence between finite points of the two spaces can be used to derive from a polynomial $f(x, y)$, its one-homogenization $F(X, Y, W)$. To do so, make the substitution $(x, y) = (X/W, Y/W)$ and clear denominators by multiplying by W^d, where d is the degree of f. For finite points, the solutions to $f = 0$ and $F = 0$ are identical, but $F^{-1}(0)$ additionally contains solutions at infinity.

2.1.2 Isotropic coordinates

For most derivations, it turns out that a linear change of coordinates to $(p, \bar{p}) = (x + iy, x - iy)$ is very convenient. For real (x, y), this just replaces the Cartesian plane with the complex plane, considering the x and y axes as the real and imaginary directions, respectively. The pair of coordinates $(p, \bar{p})$ are known as the *isotropic coordinates* of the plane. It is tempting to consider $\bar{p}$ as the complex conjugate of p, as it plays this role whenever x and y are real. However, just as we allow Cartesian coordinates x and y to take on complex values, we correspondingly must consider points where p and $\bar{p}$ are not complex conjugates [1] .

The one-homogeneous completion of the plane in isotropic coordinates, which we write as $[P, \bar{P}, W] \in \mathbb{P}^2$, follows similarly to the Cartesian case, using the correspondence $(p, \bar{p}) = (P/W, \bar{P}/W)$.

2.1.3 Isotropic points I and J

In any treatment of rigid-body motion, the concept of distance is fundamental. In Cartesian coordinates, the squared distance between points (x, y) and (a, b) is $d((x, y), (a, b)) = (x - a)^2 + (y - b)^2$.

Two real points are zero distance apart only if they are the same point. This is not true over the complexes; for example, $d((1, i), (0, 0)) = 1^2 + i^2 = 0$. In fact, the squared distance factors as

$$d((x, y), (a, b)) = [(x - a) + i(y - b)][(x - a) - i(y - b)],$$

from which one sees that there are two lines of points that are zero distance away from (a, b), namely,

$$(x - a) + i(y - b) = 0 \quad \text{and} \quad (x - a) - i(y - b) = 0.$$

In a one-homogenization of these equations, one finds that these zero-distance lines, $(X - aW) + i(Y - bW) = 0$ and $(X - aW) - i(Y - bW) = 0$, meet the

[1] Note that x and y are real if, and only if, the corresponding p and $\bar{p}$ are complex conjugates. Thus, if $p^* = \bar{p}$, where * denotes complex conjugation, we call $(p, \bar{p})$ a *real* point even though p and $\bar{p}$ may be complex.

line at infinity, $W = 0$, at one point each. These are the so-called *isotropic points*

$$I = [1, -i, 0] \quad \text{and} \quad J = [1, i, 0].$$

Due to their kinship to distance, the isotropic points have a special relationship to most of the curves studied in planar kinematics. Of particular note is the fact that *every circle passes through the isotropic points.* To see this, consider that the circle centered on (a, b) of radius r has the equation $d((x, y), (a, b)) - r^2 = 0$. When one-homogenized and intersected with $W = 0$, the radius becomes irrelevant, and we have the same situation as in the previous paragraph.

In isotropic coordinates, the squared distance between points $(p, \bar{p})$ and $(q, \bar{q})$ is

$$d((p, \bar{p}), (q, \bar{q})) = (p - q)(\bar{p} - \bar{q}),$$

which reflects the fact that a complex vector times its own conjugate gives its squared magnitude. The factorization is apparent and one sees that in isotropic coordinates, the isotropic points are

$$I = [1, 0, 0] \quad \text{and} \quad J = [0, 1, 0].$$

Thus, the isotropic coordinates are seen to result from choosing coordinate axes that align with the isotropic points. The simple form of the distance function and the isotropic points is the reason that isotropic coordinates often lead to simpler derivations than Cartesian coordinates. The remainder of this paper will use isotropic coordinates exclusively.

2.2 Two-homogenization of the plane

There is another way to compactify the plane: we may introduce a separate homogeneous coordinate for each of p and $\bar{p}$. The new coordinates are written as $([P, W], [\bar{P}, \bar{W}])$, where at least one coordinate in each of $[P, W]$ and $[\bar{P}, \bar{W}]$ must be nonzero. For all finite points, we have the correspondence $(p, \bar{p}) = (P/W, \bar{P}/\bar{W})$. The two-homogenization $F(P, W, \bar{P}, \bar{W})$ of a function $f(p, \bar{p})$ is derived by making this substitution and clearing denominators. Mathematicians call $[P, W] \in \mathbb{P}^1$ a one-dimensional complex projective space, and $([P, W], [\bar{P}, \bar{W}]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the cross product of two such spaces. Note that the two-homogenization has singled out the coordinate axes as special directions. Accordingly, it is not generally useful to two-homogenize Cartesian coordinates, but it turns out to be very useful to two-homogenize the isotropic coordinates.

The two-homogenization of the plane, $\mathbb{P}^1 \times \mathbb{P}^1$, has no effect on the geometry of finite points, but it radically alters the picture at infinity. There are now *two* lines at infinity: $W = 0$ and $\bar{W} = 0$. A line $p - q = 0$, parallel to the $\bar{p}$-axis, two-homogenizes to $P - qW = 0$. It does not meet $W = 0$, but hits $\bar{W} = 0$ in the point $([q, 1], [1, 0])$. Similarly, a line $\bar{p} - \bar{q} = 0$ meets infinity in the point $([1, 0], [\bar{q}, 1])$. All lines not parallel to a coordinate axis meet infinity in the same point: $([1, 0], [1, 0])$. If we try to associate points at infinity in the one-homogeneous treatment of the plane with those of the two-homogeneous isotropic treatment of the plane, we find that isotropic point I has been replaced by the line $W = 0$, isotropic point J has been replaced by line $\bar{W} = 0$, and all other points on the one-homogeneous line at infinity have collapsed to the single point $([1, 0], [1, 0])$. A nice illustration of this "blow up/blow down" process, which is one of the basic constructions in algebraic geometry, can be found in [5, ex.7.22].

An important consequence of the transformation into $\mathbb{P}^1 \times \mathbb{P}^1$ is that two general circles no longer meet at infinity. (In $\mathbb{P}^2$, they always meet at the isotropic points.) Instead, the two-homogenization of a circle with center $(q, \bar{q})$ and radius r,

$$(P - qW)(\bar{P} - \bar{q}\bar{W}) - r^2 W\bar{W} = 0, \tag{1}$$

hits the line $W = 0$ at $([1, 0], [\bar{q}, 1])$ and hits $\bar{W} = 0$ at $([q, 1], [1, 0])$. This shows that only concentric circles meet at infinity, which reflects the fact that they do not have finite points of intersection.

2.3 Support polynomials

Suppose that we have a polynomial

$$g(p, \bar{p}) = \sum_{(j,k)\in\mathcal{I}} \alpha_{jk} p^j \bar{p}^k,$$

where $\mathcal{I}$ is just the index set of all the terms appearing in the polynomial. Further, let $\mathcal{I}_{uv}$ denote the subset of indicies in $\mathcal{I}$ that maximize $ju + kv$, let $d_{uv} = \max_{\mathcal{I}}(ju + kv)$ and let $g_{uv}(p, \bar{p}) = \sum_{(j,k)\in\mathcal{I}_{uv}} \alpha_{jk} p^j \bar{p}^k$. The latter is called the *support polynomial* of g for direction (u, v). In particular, g_{11} consists of the terms of maximal degree $d = d_{11}$, g_{10} consists of the terms of maximal bidegree d_{10} in p, and g_{01} comprises the terms of maximal bidegree d_{01} in $\bar{p}$. (*Bidegree* means the degree in one of the variables, treating the other

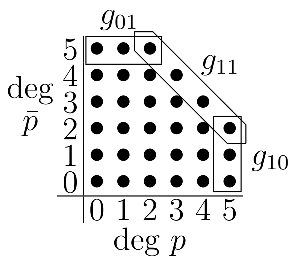


Fig. 1. Monomials (plotted by their degrees) of a curve with degree $d = 7$, bidegree $b = 5$, and circularity $c = 2$. The support polynomials g_{11}, g_{10}, and g_{01} are indicated.

variable as a constant.) These support polynomials are indicated in a plot of the monomials for a particular degree 7 polynomial in Fig.1.

For kinematics, we are interested in *real* curves. In Cartesian coordinates, these are polynomials $f(x, y)$ that have real coefficients, and hence f gives a real value whenever x and y are both real. Accordingly, the transformation of f to isotropic coordinates,

$$g(p, \bar{p}) = f((p + \bar{p})/2, (p - \bar{p})/(2i)),$$

has real values whenever p and $\bar{p}$ are complex conjugates. This implies that the coefficients in $g(p, \bar{p})$ obey the relation $a_{jk} = a^*_{kj}$, that is, terms appear in complex conjugate pairs. Consequently, $d_{10} = d_{01}$, so we may call $b = d_{10} = d_{01}$ simply the bidegree of the curve. The terms may be shown in a plot of their exponents as illustrated in Fig 1. Clearly, $d \geq b$, and we call $c = d - b \geq 0$ the *circularity* of the curve. Curves with nonzero circularity are common in planar kinematics.

Support polynomials determine the behavior of the curve at infinity. The one-homogenization $G(P, \bar{P}, W) = 0$ of curve $g(p, \bar{p}) = 0$ meets infinity at the roots of $G(P, \bar{P}, 0)$. Setting the homogeneous coordinate W to zero annihilates all but the terms of highest degree, hence

$$G(P, \bar{P}, 0) = G_{11}(P, \bar{P}, 0) = g_{11}(P, \bar{P}).$$

For bidegree b and circularity c, assuming that the coefficients $\alpha_{b,c} = \alpha^*_{c,b} \neq 0$, G_{11} contains the factor $P^c \bar{P}^c$, so we have that the curve passes through each of the isotropic points c times.

In a two-homogeneous formulation, the same curve $g(p, \bar{p}) = 0$ hits infinity in the p direction at the roots of G_{10} and in the $\bar{p}$ direction at the roots of G_{01}, each of which is a degree c polynomial. The c appearances of each of the isotropic points in the one-homogeneous formulation have each become c (generally distinct) points at infinity in the two-homogeneous treatment. There is one point on the line $W = 0$ at infinity in $\mathbb{P}^1 \times \mathbb{P}^1$ for each line through the isotropic point $I = [1, 0, 0]$ in $\mathbb{P}^2$, and a similar correspondence holds between points on $\bar{W} = 0$ and lines through $J = [0, 1, 0]$. A curve in $\mathbb{P}^2$ that passes through I along c different tangent directions will, after transformation into $\mathbb{P}^1 \times \mathbb{P}^1$, hit the line $W = 0$ in c distinct points. This separation of the tangent lines through I and J is precisely what makes the two-homogeneous formulation convenient for studying foci.

3 Foci and Singular Foci

We begin this section with the traditional definitions of a focal point and a singular focal point. These definitions may seem at the outset a bit abstruse, but reconsidering them using isotropic coordinates, we will see that the definitions become quite simple.

The traditional definition of a focal point assumes a one-homogeneous treatment of infinity, so that we may speak of the isotropic points I and J. Given an algebraic curve and a point, there will be a finite number of lines through the point that are also tangent to the curve. This fact also applies when the point in question is an isotropic point, which is crucial to the following definitions [3].

Definition 1 *A* **focal point** *of an algebraic curve in the plane is defined as the point of intersection of a tangent through isotropic point I with a tangent through isotropic point J.*

Definition 2 *A* **singular focal point** [2] *of an algebraic curve in the plane is the intersection of a tangent* at *isotropic point I with a tangent* at *isotropic point J.*

Notice that a curve can have a singular focus only if it passes through the isotropic points; that is, it must have positive circularity.

The name "singular focus" is appropriate because such a point represents the coalescence of several foci for nearby curves having the same degree but smaller circularity. Suppose a curve of degree greater than one passes near but not through a point P. Locally, the curve looks like its osculating circle, and there are two nearly parallel tangents passing through P. As the curve deforms continuously to pass through P, these two tangents coalesce into a double tangent line. In the case that P is one of the isotropic points, the foci on the the two tangents coalesce as well. It is also possible for a curve to pass

[2] Bottema and Roth [1] note that singular foci are also sometimes called *special, principal,* or *Laguerre* foci.

through an isotropic point with zero curvature, which implies a multiple root of order greater than two.

It is helpful to translate these geometric definitions into algebraic equations. Suppose that the curve is given by the equation $f(p, \bar{p}) = 0$. Let $(q, \bar{q})$ be a point on the curve, hence

$$f(q, \bar{q}) = 0. \tag{2}$$

The line tangent to the curve at $(q, \bar{q})$ is

$$f_p(q, \bar{q})(p - q) + f_{\bar{p}}(q, \bar{q})(\bar{p} - \bar{q}) = 0, \tag{3}$$

where f_p and $f_{\bar{p}}$ denote the partial derivatives of f with respect to p and $\bar{p}$, respectively. Given $(p, \bar{p})$, a simultaneous solution of Eqs.(2,3) for $(q, \bar{q})$ determines the point of tangency for a tangent through $(p, \bar{p})$. Accordingly, we may find the foci of the curve by homogenizing these equations and substituting the isotropic points for $(p, \bar{p})$. The homogenized tangent equation is

$$f_p(q, \bar{q})(P - Wq) + f_{\bar{p}}(q, \bar{q})(\bar{P} - W\bar{q}) = 0. \tag{4}$$

For isotropic point $I = [1, 0, 0]$, this becomes simply

$$f_p(q, \bar{q}) = 0. \tag{5}$$

Let $(q_I, \bar{q}_I)$ be one of the solutions to the pair of equations (2,5). When substituted back into Eq.3, this gives the equation for the tangent line as $\bar{p} = \bar{q}_I$. Similarly, points of tangency for the tangents through J are the solutions of Eq.2 with $f_{\bar{p}}(q, \bar{q}) = 0$. Denoting one such point as $(q_J, \bar{q}_J)$, the corresponding tangent line is $p = q_J$, and hence point $(q_J, \bar{q}_I)$ is a focal point. If $f(q, \bar{q}) = 0$ is a real curve, then we can obtain the real foci by pairing up each q_J with a $\bar{q}_I$ that is its complex conjugate. Due to this complex-conjugate relationship, one only needs to find the tangents through one of the isotropic points to obtain both sets of tangents.

Example 1 Foci of an ellipse. *An ellipse centered at the origin and aligned with the coordinate axes has the equation $x^2/a^2 + y^2/b^2 - 1 = 0$. In isotropic coordinates, this becomes*

$$f(p, \bar{p}) = (p + \bar{p})^2/a^2 - (p - \bar{p})^2/b^2 - 4 = 0.$$

Solving this simultaneously with $f_p(p, \bar{p}) = 0$ gives $\bar{p}_I = \pm\sqrt{a^2 - b^2}$. Similarly, the tangents through J give $p_J = \pm\sqrt{a^2 - b^2}$. These combine to give four *foci. In Cartesian coordinates, these are* $(x, y) = (\pm\sqrt{a^2 - b^2}, 0)$ *and* $(x, y) = (0, \pm i\sqrt{a^2 - b^2})$. *If* $a > b$, *the first of these gives a pair of real foci on the* x*-axis, whereas for* $a < b$, *the second formula gives the real foci on the* y*-axis.*

To determine the singular foci, we examine the tangency conditions as the point of tangency approaches an isotropic point. For the tangents at isotropic point I, the leading terms in q dominate in Eq.5, hence the $\bar{q}_I$ coordinates of the singular foci are the roots of

$$(f_p)_{10}(1, \bar{q}_I) = 0. \tag{6}$$

But $(f_p)_{10}(1, \bar{q}_I) = (f_{10})_p(1, \bar{q}_I) = bf_{10}(1, \bar{q}_I)$, since differentiation with respect to p does not change which terms have the highest degree in p. Therefore, the singular foci $(q_J, \bar{q}_I)$ are the solutions to

$$f_{10}(1, \bar{q}_I) = 0, \qquad f_{01}(q_J, 1) = 0. \tag{7}$$

Recall from the previous section that these roots are associated with the roots at infinity in a two homogenization of the curve. This leads us to the following theorem.

Theorem 1 *Let* $F(P, W, \bar{P}, \bar{W})$ *be the two homogenization of a polynomial* $f(p, \bar{p})$. *The singular focal points of* $f(p, \bar{p}) = 0$ *are the points* $(p, \bar{p}) = (q_J, \bar{q}_I)$, *where* q_J *is any root of* F *of the form* $([q_J, 1], [1, 0])$, *and* $\bar{q}_I$ *is any root of* F *of the form* $([1, 0], [\bar{q}_I, 1])$.

Example 2 Singular foci of a circle. *At Eq.(1), we determined that the points at infinity of the two-homogenization of a circle with center* $(q, \bar{q})$ *are*

$$([q, 1], [1, 0]) \quad \textit{and} \quad ([1, 0], [\bar{q}, 1]).$$

Thus, the circle has its center as its only singular focus. Note that in light of Example 1, considering the circle as a special ellipse, the center point is seen to be the coalescence of four regular foci as the semi-major and semi-minor axes, a *and* b, *become equal.*

4 Dixon Determinant

In [11], a general formulation is derived, using the Dixon determinant, for the polynomial curve traced by any planar linkage with rotational joints. The method begins by writing the loop closure equations, using complex numbers to represent vectors in the plane. In these equations, let position vector p close a loop around the tracing point. Then, for a one-degree-of-freedom mechanism having $N = 2n$ links, there are n independent loop equations. These may be written, for $k = 1, \ldots, n$ as

$$c_{k0} + \sum_{i=1}^{N-1} c_{ki}\theta_i + c_{kN}p = 0, \tag{8}$$

where c_{ki} is the vector of link i that appears in loop k, θ_i is the rotation for link i. (Note: $c_{ki} = 0$ if link i is not in loop k.) We also have the conjugate set of equations, for $k = 1, \ldots, n$,

$$c^*_{k0} + \sum_{i=1}^{N-1} c^*_{ki}\theta_i^{-1} + c^*_{kN}\bar{p} = 0. \tag{9}$$

We wish to eliminate all of the θ_i to obtain a single polynomial tracing curve equation in $(p, \bar{p})$. The main result in [11] is written for input/output polynomials, but when applied to tracing curves, it reads as follows.

Theorem 2 *[11]. The Dixon determinant for Eqs.(8,9), which is a necessary condition for them to have a common solution, can be written as*

$$f(p, \bar{p}) = \det \begin{pmatrix} D_1 p + D_2 & A^T \\ A & \sigma(D_1^*\bar{p} + D_2^*) \end{pmatrix} = 0, \tag{10}$$

where $\sigma = (-1)^{n-1}$, D_1 and D_2 are diagonal and the elements of A obey the relation $a_{ij} = \sigma a^*_{ji}$. Matrices D_1, D_2, and A are all size $m \times m$ for some $m \leq \binom{2n-1}{n}$ and each is a homogeneous polynomial function of the link parameters c_{ki}, c^*_{ki}.

The dependence of D_1, D_2, and A on the link parameters may be determined using the procedure described in [11].

Since this is just a necessary condition, it is possible that $f(p, \bar{p})$ contains an extraneous factor. But, if $f = gh$ for polynomials f, g and h, then $f_{uv} = g_{uv}h_{uv}$; that is, the support polynomial for f in the (u, v) direction is the product of those for its factors. This implies that the singular foci of f are the union of those for its factors g and h. Thus, we may find all of the singular foci of the tracing curve by finding those of f and casting out any extraneous ones.

By Eq.7, to find the singular foci, we solve the support polynomial $f_{01}(q_J, 1) = 0$ consisting of the terms of f having maximal degree in $\bar{p}$. Since a determinant is multilinear in its columns, we may find $f_{01}(p, \bar{p})$ by retaining in each column only the terms of maximal degree in $\bar{p}$. Each nonzero entry in D_1^* stands alone in its column, but in any column where D_1^* has a zero, the whole column is constant and is therefore retained. For notational convenience, assume the columns and rows are ordered so as to place the nonzero entries into submatrix D_{11} of D_1 (and similarly for D_1^*), and partition the matrices D_2 and A accordingly, to express $f_{01}(q_J, 1) = 0$ as

$$\det \begin{pmatrix} D_{11}q_J + D_{21} & \mathbf{0} & \mathbf{0} & A_{12}^T \\ \mathbf{0} & D_{22} & \mathbf{0} & A_{22}^T \\ A_{11} & A_{12} & \sigma D_{11}^* & \mathbf{0} \\ A_{21} & A_{22} & \mathbf{0} & \sigma D_{22}^* \end{pmatrix} = 0 \tag{11}$$

Only one block in the third column is nonzero, so this may be reduced to

$$\det(\sigma D_{11}^*) \det \begin{pmatrix} D_{11}q_J + D_{21} & \mathbf{0} & A_{12}^T \\ \mathbf{0} & D_{22} & A_{22}^T \\ A_{21} & A_{22} & \sigma D_{22}^* \end{pmatrix} = 0. \tag{12}$$

Numerical values for the singular foci can be obtained from this equation using eigenvalue algorithms. Symbolic expressions can be obtained this way as well.

An important special case is when all entries on the diagonal of D_1 are nonzero, so that only the upper left block of the above condition appears, the other entries being empty. Thus the condition reduces to $\det(D_1 q_J + D_2) = 0$, which implies that the singular foci are simply

$$q_J = -\mathrm{diag}(D_1^{-1} D_2), \tag{13}$$

where diag() extracts the diagonal of a matrix. Even when D_1 has some zeros on the diagonal, causing A_{21} to be present in Eq.12, a condition similar to Eq.13 applies to the subset of columns where A_{21} has all zero entries. Since A is sparse, this condition comes into play often.

Since we are only concerned with real curves, the conjugate conditions give $\bar{q}_I = q_J^*$. So we need only solve either Eq.12 or Eq.13, as appropriate, to get q_J and find $\bar{q}_I$ by conjugation.

Example 3 Singular foci of a four-bar.

Consider the four-bar linkage A_0ADBB_0, which is a sub-mechanism of the six-bar in Fig.2. Letting $p = \vec{OD}$, the loop equations are

$$\begin{aligned} a_0 + a_1\theta_1 + a_2\theta_2 - p = 0 \\ b_0 + b_2\theta_2 + b_3\theta_3 - p = 0 \end{aligned} \tag{14}$$

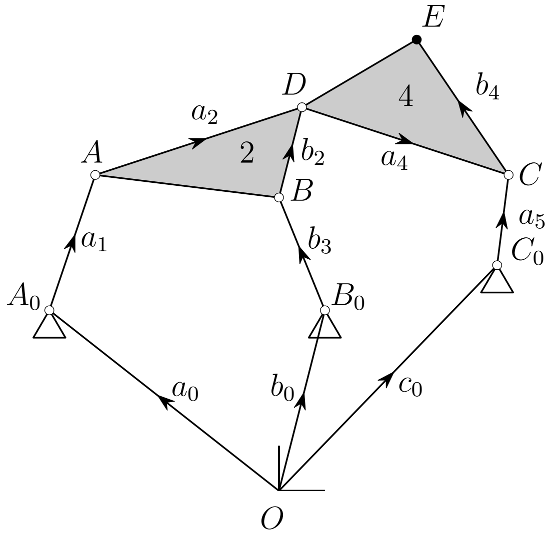


Fig. 2. Four-bar (point D) and Stephenson-3 six-bar (point E) path-generating linkages

Applying the Dixon determinant and dividing common factors from each column, we obtain the polynomial for the curve traced by coupler point D, expressed in terms of isotropic coordinates $(p, \bar{p})$ as the following determinant set to zero:

$$\left|\begin{array}{ccc|ccc} p-b_0 & 0 & 0 & 0 & \bar{a}_2\bar{b}_3 & -\bar{a}_2 \\ 0 & d & 0 & -\bar{b}_3 & 0 & -\bar{a}_1 \\ 0 & 0 & p-a_0 & -\bar{b}_2 & -\bar{a}_1\bar{b}_2 & 0 \\ \hline 0 & a_2b_3 & -a_2 & \bar{p}-\bar{b}_0 & 0 & 0 \\ -b_3 & 0 & -a_1 & 0 & \bar{d} & 0 \\ -b_2 & -a_1b_2 & 0 & 0 & 0 & \bar{p}-\bar{a}_0 \end{array}\right| = 0, \tag{15}$$

where

$$d = (p-a_0)b_2 - (p-b_0)a_2, \quad \bar{d} = (\bar{p}-\bar{a}_0)\bar{b}_2 - (\bar{p}-\bar{b}_0)\bar{a}_2.$$

Equation 13 applies, so the foci are

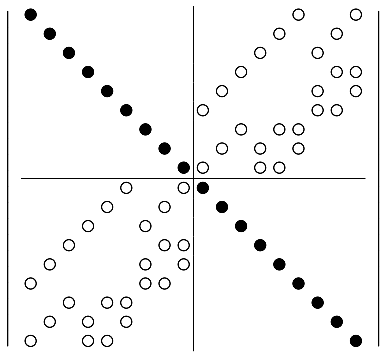
$$q_J = \{b_0, (a_0b_2 - b_0a_2)/(b_2 - a_2), a_0\}.$$

Two of these are the two fixed pivots A_0 and B_0, and the third is the vertex of a triangle similar to the coupler triangle built on the fixed pivots. This is, of course, a well-known result.

Example 4 Singular foci of the Stephenson-3 six bar. *Consider the six-bar linkage shown in Fig.2, formed by adding a dyad, links 4 and 5, to the four-bar linkage of the previous example. Letting $p = \vec{OE}$, the loop equations are*

$$\begin{aligned}(a_0 - b_0) + a_1\theta_1 + (a_2 - b_2)\theta_2 - b_3\theta_3 &= 0\\ (a_0 - c_0) + a_1\theta_1 + a_2\theta_2 + a_4\theta_4 - a_5\theta_5 &= 0\\ c_0 + b_4\theta_4 + a_5\theta_5 - p &= 0\end{aligned} \tag{16}$$

Applying the Dixon determinant, one obtains an 18×18 matrix with the following sparsity pattern, in which a closed circle denotes an entry containing p or $\bar{p}$, and an open circle denotes a constant entry:



$= 0.$

Hence, Eq.13 applies and gives 9 focal points. Three of these are the focal triangle of the base four-bar, as elucidated above, three are repetitions of the other ground pivot, $(c_0, \bar{c}_0)$, and the final three are a stretch-rotation of the focal triangle about $(c_0, \bar{c}_0)$. Further discussion of this example appears in §6.

It was mentioned above that the Dixon determinant may introduce an extraneous factor. This factor's singular foci may be distinct from those of the tracing curve or may coincide making the focus appear to have extra multiplicity. The valid foci and their correct multiplicities can be determined as follows. Suppose that point $(q, \bar{q})$ is to be tested as a singular focus. Intersect the tracing curve with a circle of arbitrary radius, say r, centered at $(q, \bar{q})$. Viewed two-homogeneously, if $(q, \bar{q})$ is a singular focus, the circle and the tracing curve share a point on each of the two lines at infinity, and hence the number of finite intersections is reduced by two times the multiplicity of the focus. These computations can be done by simply substituting

$$p = q + r\phi, \qquad \bar{p} = \bar{q} + r\phi^{-1}$$

into Eq.10 and solving for ϕ using a generalized eigenvalue routine. Eigenvalues of $\phi = 0$ or $\phi^{-1} = 0$ may occur, but these must be dropped. At each of the

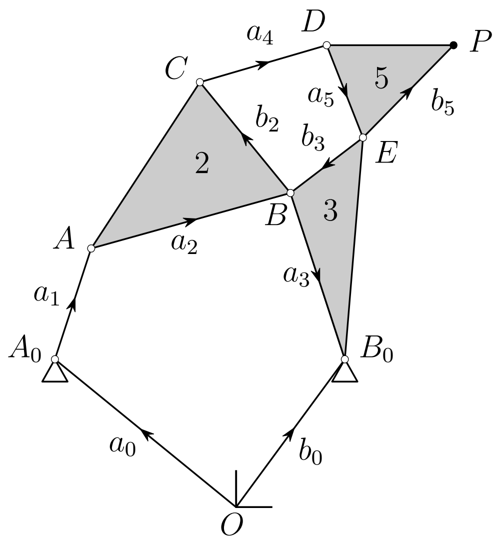


Fig. 3. Watt-1 path-generating linkage

remaining finite, nonzero solutions, back substitute to find all the joint values (see [11] for details) and check these in the original loop equations. Let m_q be the number of valid solutions obtained in this way and let m be the number for an arbitrary test point. [3] Then, $(m - m_q)/2$ is the multiplicity of $(q, \bar{q})$ as a singular focus. A multiplicity of zero indicates that $(q, \bar{q})$ is not a singular focus. The following example illustrates the use of this technique.

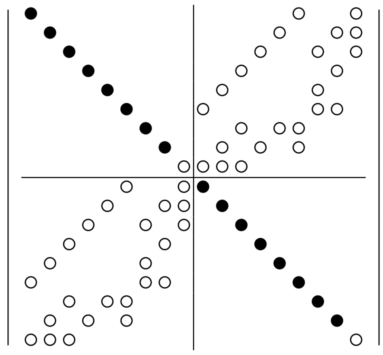
Example 5 Singular foci of a Watt-1 six bar. *Consider the Watt six-bar linkage shown in Fig.3. Denoting* $p = \vec{OP}$*, the loop equations are*

$$\begin{aligned}
(a_0 - b_0) + a_1\theta_1 + a_2\theta_2 + a_3\theta_3 &= 0 \\
b_2\theta_2 + b_3\theta_3 + a_4\theta_4 + a_5\theta_5 &= 0 \\
b_0 - (a_3 + b_3)\theta_3 + b_5\theta_5 - p &= 0
\end{aligned} \tag{17}$$

Applying the Dixon determinant, one obtains an 18×18 *matrix with the following sparsity pattern, in which a closed circle denotes an entry containing* p

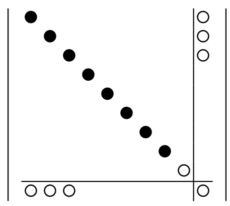
[3] If either test gives solutions with multiplicity greater than one, a higher-order analysis may be required to determine m or m_q.

or $\bar{p}$, and an open circle denotes a constant entry:



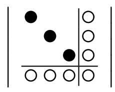
$= 0.$

Because the final entry in D_1 is zero, Eq.12 must be used. This results in a matrix with the following pattern, composed of the original upper left block above and only the last row and column of the rest:



$= 0.$

The fourth through eighth diagonal entries give focal equations directly, in the style of Eq.13, leaving an eigenvalue problem of the form



$= 0.$

This gives three focal points. Among the combined eight focal points, $q_J = b_0$ (that is, ground pivot B_0 in Fig.3) appears four times. However, checking the intersections of the tracing curve with a circle centered on B_0 reveals that B_0 is only a triple singular focus. The Dixon determinant is thus seen to include a bilinear extraneous factor that has a singular focus $(b_0, \bar{b}_0)$. It turns out that the extraneous factor is just $(p - b_0)(\bar{p} - \bar{b}_0)$, which can be divided out of the Dixon determinant to get precisely the tracing curve equation. In the final tally, counting multiplicities, there are seven foci, as is consistent with the known result that this tracing curve is fully circular of degree 14. Calculations supporting these statements are presented in §6.

5 Sliding Joints

The analysis of the previous section assumed all rotational joints. If one or more joints is a sliding joint, some adjustments must be made, but the overall procedure will be similar. One can still form the Dixon determinant, and by Theorem 1, evaluate the singular foci by finding the roots for the terms of maximal degree in $\bar{p}$. The Dixon matrix will generally be smaller than that obtained when the slider joint is replaced by a rotational one, but some of the entries will now involve both p and $\bar{p}$, instead of being entirely separated as in the all-rotational case. The upshot is that the tracing curve is no longer fully circular, meaning that, similar to Fig. 1, there is more than one term of maximal degree. So, in addition to the singular foci, the roots of the support polynomial in the (1,1) direction will be of interest.

6 Numerical Examples

To illustrate the method, we present numerical results for the six-bar linkages in Figs. 2,3.

For the Stephenson-3 example of Fig. 2, the link parameters are

$$\begin{aligned} a_0 &= -1.0 + 0.8\mathrm{i}, & b_0 &= \ \ 0.2 + 0.8\mathrm{i}, \\ c_0 &= \ \ 0.95 + 1.0\mathrm{i}, & a_1 &= \ \ 0.2 + 0.6\mathrm{i}, \\ a_2 &= \ \ \ 0.9 + 0.3\mathrm{i}, & b_2 &= \ \ 0.1 + 0.4\mathrm{i}, \\ b_3 &= -0.2 + 0.5\mathrm{i}, & a_4 &= \ \ 0.9 - 0.3\mathrm{i}, \\ b_4 &= -0.4 + 0.6\mathrm{i}, & a_5 &= 0.05 + 0.4\mathrm{i}. \end{aligned} \tag{18}$$

The resulting nine values of the singular foci are

$$F = \left[\begin{pmatrix} 0.9500 + 1.0000\mathrm{i} \\ 0.9500 + 1.0000\mathrm{i} \\ 0.9500 + 1.0000\mathrm{i} \\ 0.4067 + 1.2300\mathrm{i} \\ 0.7353 + 1.5611\mathrm{i} \end{pmatrix} \begin{pmatrix} -0.3133 + 1.7900\mathrm{i} \\ 0.2000 + 0.8000\mathrm{i} \\ 0.2738 + 1.4092\mathrm{i} \\ -1.0000 + 0.8000\mathrm{i} \\ \ \end{pmatrix} \right]. \tag{19}$$

We observe that ground pivots $A_0 = F_9$, $B_0 = F_7$, and $C_0 = F_1 = F_2 = F_3$ are all singular foci, with C_0 appearing with multiplicity 3. One may verify that F_8 is the third singular focus of the four-bar submechanism A_0ADBB_0, as derived in Ex.3. Moreover, one may also verify that

$$(F_4, F_5, F_6) = c_0 - b_4/a_4[(F_7, F_8, F_9) - c_0], \tag{20}$$

that is, the points (F_4, F_5, F_6) are a triangle similar to the focal triangle of the four-bar sub-mechanism, specifically, a stretch-rotation by factor $(-b_4/a_4)$ around point c_0.

The relation given in Eq.(20) holds in general, as can be concluded from the construction given in [8] for path cognates for the Stephenson-3 six-bar. A path cognate in Roth's construction contains a four-bar sub-mechanism that is the same stretch-rotation about point c_0. Since the new four-bar in the cognate plays the same role as the original four-bar sub-mechanism, its singular foci must also be singular foci of the six-bar.

For the Watt-1 six-bar of Fig. 3, the link parameters are

$$\begin{aligned}
a_0 &= -1.0 + 0.8\mathrm{i}, & b_0 &= 0.6 + 0.8\mathrm{i}, \\
a_1 &= 0.2 + 0.6\mathrm{i}, & a_2 &= 1.1 + 0.3\mathrm{i}, \\
a_3 &= 0.3 - 0.9\mathrm{i}, & b_2 &= -0.5 + 0.6\mathrm{i}, \\
a_4 &= 0.7 + 0.2\mathrm{i}, & a_5 &= 0.2 - 0.5\mathrm{i}, \\
b_3 &= -0.4 - 0.3\mathrm{i}, & b_5 &= 0.5 + 0.5\mathrm{i}.
\end{aligned} \tag{21}$$

Following the procedure given in Ex.5, we obtain from the fourth through eighth diagonal elements the values

$$F_{4,\ldots,8} = \begin{pmatrix} 0.6000 + 0.8000\mathrm{i} \\ 0.6000 + 0.8000\mathrm{i} \\ -1.2667 + 1.6000\mathrm{i} \\ 0.6000 + 0.8000\mathrm{i} \\ 1.5676 + 1.8653\mathrm{i} \end{pmatrix}. \tag{22}$$

We observe that ground pivot B_0 appears three times. Now, we must form an eigenvalue problem from rows 1,2,3,18 and columns 1,2,3,18, to get an additional three points

$$F_{1,2,3} = \begin{pmatrix} 0.6000 + 0.8000\mathrm{i} \\ -0.9495 + 1.0577\mathrm{i} \\ -0.6962 + 1.5576\mathrm{i} \end{pmatrix}. \tag{23}$$

Notice that B_0 appears a fourth time here. We have eight singular foci for the Dixon determinant, but it is known that the Watt-I linkage only has degree 14, and hence only 7 singular foci. This implies that the Dixon determinant, which is just a necessary condition, contains an extraneous factor that contributes an extra focus. Section 4 describes how to test the multiplicity of a singular focus. Any of the foci could be the culprit, but since B_0 appears multiple times, we test it first. This is done by setting $p = b_0 + r\phi$ and $\bar{p} = b_0^* + r\phi^{-1}$ for a random radius r and solving for ϕ. Letting $r = 0.8273$, we obtain eight finite, nonzero solutions for ϕ, namely

$$\phi = \begin{pmatrix} 0.03909779074084 - 0.99923538906465\mathrm{i} \\ -0.27002515994004 - 0.96285326659848\mathrm{i} \\ -0.03694001960119 + 0.99931748456227\mathrm{i} \\ -0.52557141900738 + 0.85074948341011\mathrm{i} \\ -0.72119118157503 + 0.69273608222642\mathrm{i} \\ -0.90737931035511 - 0.42031272540750\mathrm{i} \\ -0.99998074595015 + 0.00620545961142\mathrm{i} \\ -0.99907549225438 + 0.04299024048165\mathrm{i} \end{pmatrix}. \tag{24}$$

These all have magnitude $|\phi_i| = 1$, so they happen to be real. But what is important is the number of roots. If B_0 were an arbitrary point, we would get 14 roots, but instead, we get only 8, which implies that B_0 is a singular focus of multiplicity $(14-8)/2 = 3$. Hence, one of the four instances of B_0 in the list of singular foci of the Dixon determinant is due to an extraneous factor. The bottom line is that the mechanism has five distinct singular foci, and one of them, B_0, has multiplicity 3, giving the expected total of $1+1+1+1+3 = 7$ singular foci.

7 Conclusion

This paper gives a method for finding the singular foci of planar linkages, based on a formulation of the tracing curve equation using the Dixon determinant. The method is easy to automate as a numerical algorithm to handle any planar linkage having revolute joints. The approach is extensible to sliding joints, although that has not been pursued here. In simple cases, the singular foci can be read off directly from the diagonal of the Dixon matrix, but some cases require the solution of an eigenvalue problem.

The singular foci represent essential characteristics of the curve traced out by a planar linkage; in particular, they describe the behavior of the curve at infinity. Two curves sharing one or more singular foci have a reduced number of intersection points and two linkages can generate the same tracing curve only if they have all singular foci in common. These facts can be useful in the design of path-generating linkages.

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